GreatHearts®

Pre-Calculus: Week of March 30, 2020

Time Allotment: 40 minutes per day

Student Name: ________________________________

Teacher Name: Mrs. Melisa Walters

Packet Overview

Date	Objective(s)	Page Number
Monday, March 30	Eliminate one variable at a time to achieve upper triangular form,	2
Tuesday, March 31	Identify Inconsistent Systems of Equations Containing Three Variables	6
Wednesday, April 1	Evaluate the system in order to identify if it is an inconsistent system of three equations in three variables	8
Thursday, April 2	Apply substitution method to solve a System of Nonlinear Equations	10
Friday, April 3	Calculate the possible outcomes when solving a system of equations representing a circle and a line.	14

Additional Notes: We miss you all very much!

I hope that you are all staying safe and keeping that positive attitude that you always bring with you to class. Any questions? Email me directly: melisa.walters@greatheartsnorthernoaks.org

Each lesson will end with a set of math problems pertaining to that particular lesson of the day.

Please create an "Exercise Packet" which is to include all your work and completion of these daily exercises. Each day is to have a title with the date followed by the name of the lesson. Please include a title page and staple all the completed exercises. **At a later point, we will ask you to turn your exercise packet. Do not worry right now about whether that will be online or in person, simply do the problem set as I instruct with the proper titles and labels.**

Thank you for your hard work students. I appreciate all of you. Have a great day!

Academic Honesty

I certify that I completed this assignment independently in accordance with the GHNO Academy Honor Code.

Student signature:

____

I certify that my student completed this assignment independently in accordance with the GHNO Academy Honor Code.

Parent signature:

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Monday, March 30

Pre-Calculus Unit: Chapter 9
Lesson 1: Systems of Linear Equations: Three variables

Objective: Eliminate one variable at a time to achieve upper triangular form,

NOTE: I will be providing some examples specific to this lesson. Please highlight the key points as you read through this entire lesson or take notes in the margin. At the end of the lesson, the list of questions in the exercise are to be completed to ensure mastery.

Preparing for this section:
In order to solve systems of equations in three variables, known as three-by-three systems, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution (x, y, z), which we call an **ordered triple**. A system in upper triangular form looks like the following:

$$\begin{aligned} Ax + By + Cz &= D \\ Ey + Fz &= G \\ Hz &= K \end{aligned}$$

The third equation can be solved for z, and then we back-substitute to find y and x.

To write the system in upper triangular form, we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

The solution set for to a three-by-three system is an ordered triple {(x, y, z)}. Graphically, the ordered triple defines the point that is the intersection of three planes in space.

You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represent the intersection of three planes.

number of possible solutions

Figure 2 and **Figure 3** illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a **solution set** consisting of an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.

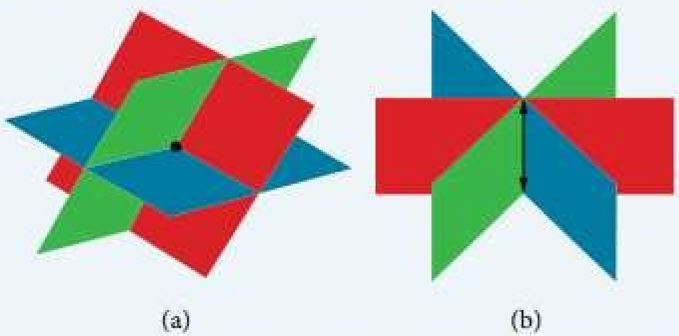


Figure 2 (a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.

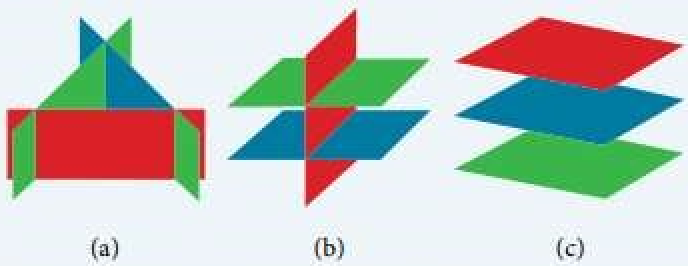


Figure 3 All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.

Example: Determining Whether an Ordered Triple Is a Solution to a System
Determine whether the ordered triple (3, −2, 1) is a solution to the system.

$$\begin{aligned} x + y + z &= 2 \\ 6x - 4y + 5z &= 31 \\ 5x + 2y + 2z &= 13 \end{aligned}$$

Solution: We will check each equation by substituting in the values of the ordered triple for x, y, and z.

$$\begin{aligned} x + y + z &= 2 \\ (3) + (-2) + (1) &= 2 \\ &\text{True} \end{aligned}$$

$$\begin{aligned} 6x - 4y + 5z &= 31 \\ 6(3) - 4(-2) + 5(1) &= 31 \\ 18 + 8 + 5 &= 31 \\ &\text{True} \end{aligned}$$

$$\begin{aligned} 5x + 2y + 2z &= 13 \\ 5(3) + 2(-2) + 2(1) &= 13 \\ 15 - 4 + 2 &= 13 \\ &\text{True} \end{aligned}$$

The ordered triple $(3, -2, 1)$ is indeed a solution to the system.

Given a linear system of three equations, **steps to solve for three unknowns**.

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Example: Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:

$$\begin{aligned} x - 2y + 3z &= 9 \quad &(1) \\ -x + 3y - z &= -6 \quad &(2) \\ 2x - 5y + 5z &= 17 \quad &(3) \end{aligned}$$

Solution: There will always be several choices as to where to begin, but the most obvious first step here is to eliminate x by adding equations (1) and (2).

$$\begin{aligned} x - 2y + 3z &= 9 \quad &(1) \\ -x + 3y - z &= -6 \quad &(2) \\ \hline y + 2z &= 3 \quad &(3) \end{aligned}$$

The second step is multiplying equation (1) by -2 and adding the result to equation (3). These two steps will eliminate the variable x.

$$\begin{aligned} -2x + 4y - 6z &= -18 \quad &(1) \text{ multiplied by } -2 \\ 2x - 5y + 5z &= 17 \quad &(3) \\ \hline -y - z &= -1 \quad &(5) \end{aligned}$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for z by adding the two equations.

$$\begin{aligned} y + 2z &= 3 \quad &(4) \\ -y - z &= -1 \quad &(5) \\ \hline z &= 2 \quad &(6) \end{aligned}$$

Choosing one equation from each new system, we obtain the upper triangular form:

$$\begin{aligned} x - 2y + 3z &= 9 \quad &(1) \\ y + 2z &= 3 \quad &(4) \\ z &= 2 \quad &(6) \end{aligned}$$

Next, we back-substitute $z = 2$ into equation (4) and solve for y.

$$y + 2(2) = 3$$
$$y + 4 = 3$$
$$y = -1$$

Finally, we can back-substitute $z = 2$ and $y = -1$ into equation (1). This will yield the solution for x.

$$x - 2(-1) + 3(2) = 9$$
$$x + 2 + 6 = 9$$
$$x = 1$$

The solution is the ordered triple (1, −1, 2). See **Figure 4**.

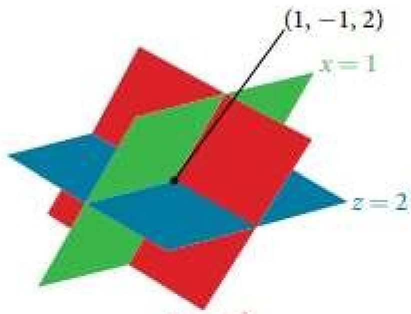


Figure 4

Directions for completing the exercises:

1. Create a packet for completed exercises. I want you staple together several papers with a title page "Exercises for Pre-Calculus"
2. At the top of each exercise completion page, title as such: Exercises for Monday, March 30th solving linear equations.
3. After you finish the entire exercise, please check your answers with the answer key at the end of the packet and attempt correction.

Exercises for Monday, March 30, 2020:

1. Solve the system of equations in three variables.

$$\begin{aligned} 2x + y - 2z &= -1 \\ 3x - 3y - z &= 5 \\ x - 2y + 3z &= 6 \end{aligned}$$

2. Determine whether the ordered triple given is the solution to the system of equations.

$$\begin{aligned} 6x - y + 3z &= 6 \\ 3x + 5y + 2z &= 0 \text{ and } (3, -3, -5) \\ x + y &= 0 \end{aligned}$$

3. Solve the system by Gaussian elimination.

$$\begin{aligned} x + y + z &= 14 \\ 2y + 3z &= -14 \\ -16y - 24z &= -112 \end{aligned}$$

4. Solve each system by Gaussian elimination.

$$\begin{aligned} x + y + z &= 0 \\ 2x - y + 3z &= 0 \\ x - z &= 1 \end{aligned}$$

Any questions right now? Please email me at melisa.walters@greatheartsnorthernoaks.org

Tuesday, March 31
Pre-Calculus: chapter 9
Lesson 2: Identifying Inconsistent Systems of Equations Containing Three Variables

Objective: Identify Inconsistent Systems of Equations Containing Three Variables

NOTE: I will be providing some examples specific to this lesson. Please highlight the key points as you read through this entire lesson or take notes in the margin. At the end of the lesson, the list of questions in the exercise are to be completed to ensure mastery.

Lesson 2
Just as with systems of equations in two variables, we may come across an **inconsistent system** of equations in three variables, which means that it does not have a solution that satisfies all three equations.

The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
The process of elimination will result in a false statement, such as 3 = 7 or some other contradiction.

Example: Solving an Inconsistent System of Three Equations in Three Variables
Solve the following system.

$$\begin{aligned} x - 3y + z &= 4 \quad (1) \\ -x + 2y - 5z &= 3 \quad (2) \\ 5x - 13y + 13z &= 8 \quad (3) \end{aligned}$$

Solution: Looking at the coefficients of x, we can see that we can eliminate x by adding equation (1) to equation (2).

$$\begin{array}{rl} x - 3y + z = 4 & (1) \\ -x + 2y - 5z = 3 & (2) \\ \hline -y - 4z = 7 & (4) \end{array}$$

Next, we multiply equation (1) by −5 and add it to equation (3).

$$\begin{array}{rl} -5x + 15y - 5z = -20 & \text{(1) multiplied by } -5 \\ 5x - 13y + 13z = 8 & (3) \\ \hline 2y + 8z = -12 & (5) \end{array}$$

Then, we multiply equation (4) by 2 and add it to equation (5).

$$\begin{array}{rl} -2y - 8z = 14 & \text{(4) multiplied by 2} \\ 2y + 8z = -12 & (5) \\ \hline 0 = 2 & \end{array}$$

The final equation 0 = 2 is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

Analysis: In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

Steps to solve using the substitution method, given a system of two equations in two variables.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Exercise for March 31, 2020

Solve the system of three equations in three variables.

$$x + y + z = 2$$
$$y - 3z = 1$$
$$2x + y + 5z = 0$$

Any questions right now? Please email me at melisa.walters@greatheartsnorthernoaks.org

<u>Wednesday, April 1, 2020</u>
Pre-Calculus: chapter 9
Lesson 3: Expressing the Solution of a System of Dependent Equations Containing Three Variables

Objective: Evaluate the system in order to identify if it is an inconsistent system of three equations in three variables

NOTE: I will be providing some examples specific to this lesson. Please highlight the key points as you read through this entire lesson or take notes in the margin. At the end of the lesson, the list of questions in the exercise are to be completed to ensure mastery.

Lesson 3:
We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different, but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

Example: Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.

$$2x + y - 3z = 0 \qquad (1)$$
$$4x + 2y - 6z = 0 \qquad (2)$$
$$x - y + z = 0 \qquad (3)$$

Solution: First, we can multiply equation (1) by −2 and add it to equation (2).

$$\begin{array}{rl} -4x - 2y + 6z = 0 & \text{equation (1) multiplied by } -2 \\ 4x + 2y - 6z = 0 & (2) \\ \hline 0 = 0 & \end{array}$$

We do not need to proceed any further. **The result we get is an identity, 0 = 0, which tells us that this system has an infinite number of solutions.**
There are other ways to begin to solve this system, such as multiplying equation (3) by −2, and adding it to equation (1). We then perform the same steps as above and find the same result, 0 = 0. When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have

$$2x + y - 3z = 0$$
$$x - y + z = 0$$
$$3x - 2z = 0$$

We then solve the resulting equation for z.

$$3x - 2z = 0$$
$$z = \frac{3}{2}x$$

We back-substitute the expression for z into one of the equations and solve for y.

$$2x + y - 3\left(\frac{3}{2}x\right) = 0$$
$$2x + y - \frac{9}{2}x = 0$$
$$y = \frac{9}{2}x - 2x$$
$$y = \frac{5}{2}x$$

So the general solution is $\left(x, \frac{5}{2}x, \frac{3}{2}x\right)$. In this solution, x can be any real number. The values of y and z are depend on the value selected for x.

Analysis: As shown in Figure 5, two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.

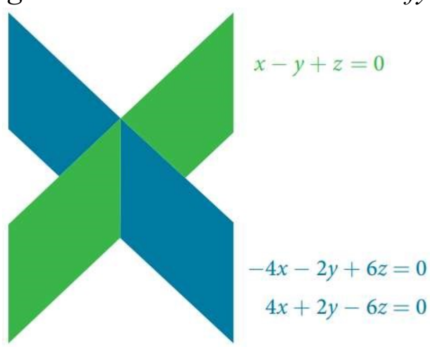


Figure 5

Question: Does the generic solution to a dependent system always have to be written in terms of x?
Answer: No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of x and if needed x and y.

Exercise for April 1, 2020:

Solve the following system.

$$x + y + z = 7$$
$$3x - 2y - z = 4$$
$$x + 6y + 5z = 24$$

Any questions right now? Please email me at melisa.walters@greatheartsnorthernoaks.org

<u>Thursday, April 2</u>

Pre-Calculus: chapter 9
Lesson 4: Systems of Nonlinear Equations and Inequalities: Two Variables

Objective: Solving a System of Nonlinear Equations Using Substitution

NOTE: I will be providing some examples specific to this lesson. Please highlight the key points as you read through this entire lesson or take notes in the margin. At the end of the lesson, the list of questions in the exercise are to be completed to ensure mastery.

Lesson 4
Halley's Comet orbits the sun about once every 75 years. Its path can be considered to be a very elongated ellipse. Other comets follow similar paths in space. These orbital paths can be studied using systems of equations. These systems, however, are different from the ones we considered in the previous section because the equations are not linear.

Halley's Comet (credit: "NASA Blueshift"/Flickr)

In this section, we will consider the intersection of a parabola and a line, a circle and a line, and a circle and an ellipse. The methods for solving systems of nonlinear equations are similar to those for linear equations.

Solving a System of Nonlinear Equations Using Substitution

A **system of nonlinear equations** is a system of two or more equations in two or more variables containing at least one equation that is not linear.

**Recall that a linear equation can take the form Ax + By + C = 0.

Any equation that cannot be written in this form in nonlinear. The substitution method we used for linear systems is the same method we will use for nonlinear systems. We solve one equation for one variable and then substitute the result into the second equation to solve for another variable, and so on. There is, however, a variation in the possible outcomes.

There are three possible types of solutions for a system of nonlinear equations involving a parabola and a line

possible types of solutions for points of intersection of a parabola and a line

Figure 2 illustrates possible solution sets for a system of equations involving a parabola and a line.

- No solution. The line will never intersect the parabola.
- One solution. The line is tangent to the parabola and intersects the parabola at exactly one point.
- Two solutions. The line crosses on the inside of the parabola and intersects the parabola at two points.

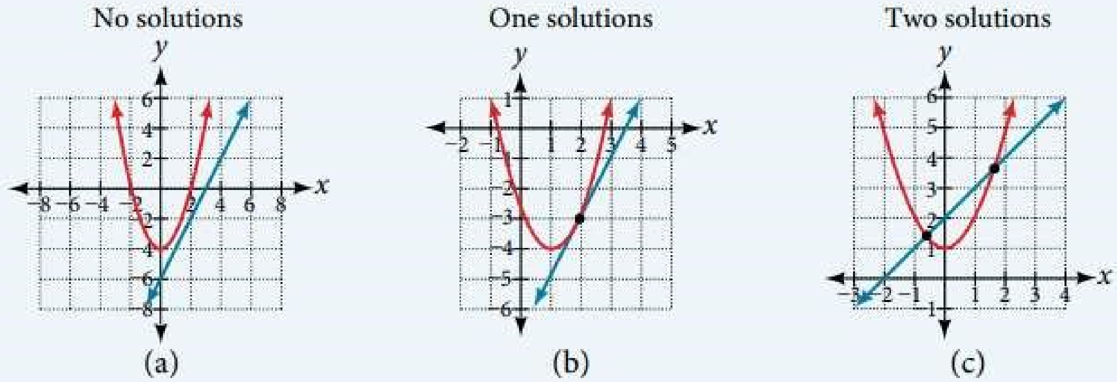


Figure 2

Steps on how to find the solution:
Given a system of equations containing a line and a parabola, find the solution.

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the parabola equation.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

Example: Solving a System of Nonlinear Equations Representing a Parabola and a Line
Solve the system of equations.

$$\begin{aligned} x - y &= -1 \\ y &= x^2 + 1 \end{aligned}$$

Solution: Solve the first equation for x and then substitute the resulting expression into the second equation.

$$\begin{aligned} x - y &= -1 && \\ x &= y - 1 && \text{Solve for } x. \\ y &= x^2 + 1 && \\ y &= (y-1)^2 + 1 && \text{Substitute expression for } x. \end{aligned}$$

Expand the equation and set it equal to zero.

$$\begin{aligned} y &= (y-1)^2 \\ &= (y^2 - 2y + 1) + 1 \\ &= y^2 - 2y + 2 \\ 0 &= y^2 - 3y + 2 \\ &= (y-2)(y-1) \end{aligned}$$

Solving for y gives $y = 2$ and $y = 1$. Next, substitute each value for y into the first equation to solve for x. Always substitute the value into the linear equation to check for extraneous solutions.

$$\begin{aligned} x - y &= -1 \\ x - (2) &= -1 \\ x &= 1 \\ x - (1) &= -1 \\ x &= 0 \end{aligned}$$

The solutions are (1, 2) and (0, 1), which can be verified by substituting these (x, y) values into both of the original equations. See **Figure 3**.

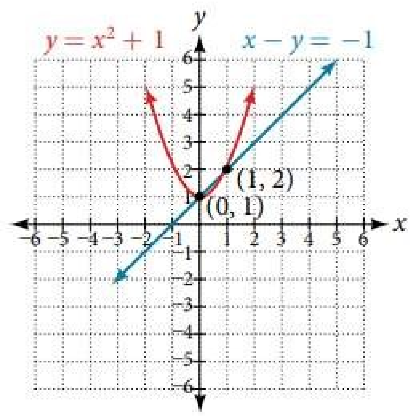


Figure 3

Question: Could we have substituted values for y into the second equation to solve for x in Example 1?

Answer: Yes, but because x is squared in the second equation this could give us extraneous solutions for x.
For y = 1

$$\begin{aligned} y &= x^2 + 1 \\ 1 &= x^2 + 1 \\ x^2 &= 0 \\ x &= \pm\sqrt{0} = 0 \end{aligned}$$

This gives us the same value as in the solution.
For y = 2

$$y = x^2 + 1$$
$$2 = x^2 + 1$$
$$x^2 = 1$$
$$x = \pm\sqrt{1} = \pm 1$$

Notice that −1 is an extraneous solution.

Exercise for Thursday, April 2, 2020

1. Solve the given system of equations by substitution.
$$3x - y = -2$$
$$2x^2 - y = 0$$

2. solve the system of nonlinear equations using substitution.
$$y = x - 3$$
$$x^2 + y^2 = 9$$

3. solve the system of nonlinear equations using substitution.
$$y = -x$$
$$x^2 + y^2 = 9$$

Any questions right now? Please email me at melisa.walters@greatheartsnorthernoaks.org

Friday, March 27

Pre-Calculus: chapter 9
Lesson 5: Intersection of a Circle and a Line

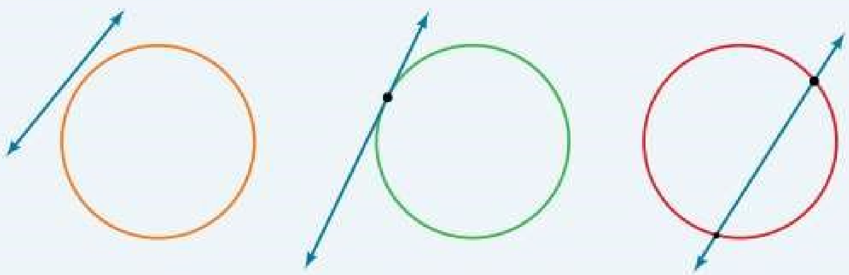
Objective: Calculate the possible outcomes when solving a system of equations representing a circle and a line.

Lesson 5
Just as with a parabola and a line, there are three possible outcomes when solving a system of equations representing a circle and a line.

possible types of solutions for the points of intersection of a circle and a line

Figure 4 illustrates possible solution sets for a system of equations involving a circle and a line.

- No solution. The line does not intersect the circle.
- One solution. The line is tangent to the circle and intersects the circle at exactly one point.
- Two solutions. The line crosses the circle and intersects it at two points.



No solutions One solution Two solutions

Figure 4

Steps to find a solution:
Given a system of equations containing a line and a circle, find the solution.

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the equation for the circle.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

Example: Finding the Intersection of a Circle and a Line by Substitution
Find the intersection of the given circle and the given line by substitution.

$$x^2 + y^2 = 5$$
$$y = 3x - 5$$

Solution: One of the equations has already been solved for y. We will substitute $y = 3x - 5$ into the equation for the circle.

$$x^2 + (3x - 5)^2 = 5$$
$$x^2 + 9x^2 - 30x + 25 = 5$$
$$10x^2 - 30x + 20 = 0$$

Now, we factor and solve for x.

$$10(x^2 - 3x + 2) = 0$$
$$10(x - 2)(x - 1) = 0$$
$$x = 2$$
$$x = 1$$

Substitute the two x-values into the original linear equation to solve for y.

$$y = 3(2) - 5$$
$$= 1$$
$$y = 3(1) - 5$$
$$= -2$$

The line intersects the circle at (2, 1) and (1, −2), which can be verified by substituting these (x, y) values into both of the original equations. See **Figure 5**.

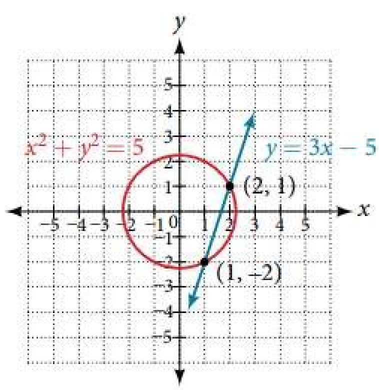


Figure 5

Exercise for April 3, 2020
Solve the system of nonlinear equations.

$$x^2 + y^2 = 10$$
$$x - 3y = -10$$

Any questions right now? Please email me at melisa.walters@greatheartsnorthernoaks.org

ANSWER KEY

Exercises for Monday, March 30, 2020:

1. Solve the system of equations in three variables.

$$2x + y - 2z = -1$$
$$3x - 3y - z = 5$$
$$x - 2y + 3z = 6$$

Answer: (1, −1, 1)

2. Determine whether the ordered triple given is the solution to the system of equations.

$$6x - y + 3z = 6$$
$$3x + 5y + 2z = 0 \text{ and } (3, -3, -5)$$
$$x + y = 0$$

Answer: NO

3. Solve the system by Gaussian elimination.

$$x + y + z = 14$$
$$2y + 3z = -14$$
$$-16y - 24z = -112$$

Answer: No solutions exist

4. Solve each system by Gaussian elimination.

$$x + y + z = 0$$
$$2x - y + 3z = 0$$
$$x - z = 1$$

Answer: $\left(\frac{4}{7}, -\frac{1}{7}, -\frac{3}{7}\right)$

Exercises for March 31, 2020

Solve the following system.

$$x + y + z = 2$$
$$y - 3z = 1$$
$$2x + y + 5z = 0$$

Answer: NO Solution

Exercises for April 1, 2020

Solve the following system.

$$x + y + z = 7$$
$$3x - 2y - z = 4$$
$$x + 6y + 5z = 24$$

Answer: Infinite number of solutions of the form $(x, 4x - 11, -5x + 18)$

Exercises for Thursday, April 2, 2020

1. Solve the given system of equations by substitution.

$$3x - y = -2$$
$$2x^2 - y = 0$$

Answer: $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(2, 8)$

2. solve the system of nonlinear equations using substitution.

$$y = x - 3$$
$$x^2 + y^2 = 9$$

Answer: $(0, -3), (3, 0)$

3. solve the system of nonlinear equations using substitution.

$$y = -x$$
$$x^2 + y^2 = 9$$

Answer: $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right), \left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$

Exercise for Friday, April 3, 2020

Solve the system of nonlinear equations.

$$x^2 + y^2 = 10$$
$$x - 3y = -10$$

Answer: $(-1, 3)$

NOTE: **At a later point, we will ask you to turn your exercise packet. Do not worry right now about whether that will be online or in person, simply do the problems as I instruct with the proper titles and labels.**