Calculus I

April 14 - April 17

Time Allotment: 40 minutes per day

Student Name: ________________________________

Teacher Name: ________________________________

Calculus
April 14-17

GreatHearts

Packet Overview

Date	Objective(s)	Page Number
Monday, April 13	Off	
Tuesday, April 14	1. Warmup and Quiz	2
Wednesday, April 15	1. Finding Volume Using Rectangle and Square Cross Sections	3
Thursday, April 16	1. Finding Volume Using Semi-Circle Cross Sections	4
Friday, April 17	1. Finding Volume Using Equilateral Triangle Cross Sections	5

Additional Notes: Dear Students: this week we're going to use integrals to enter into the third dimension and find the volumes of solids! Get excited!

Though not required to complete these assignments, Khan Academy's AP Calculus AB series of videos are a helpful resource for supplemental learning. We'll be going over the chapter on Applications of Integrals. If you're having trouble seeing the 3-D pictures on my handwritten notes, they follow closely the videos titled "Volumes with cross sections" in that Applications of Integrals section. Take a look at those while you work through the examples.

Academic Honesty

I certify that I completed this assignment independently in accordance with the GHNO Academy Honor Code.

Student signature:

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I certify that my student completed this assignment independently in accordance with the GHNO Academy Honor Code.

Parent signature:

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Calculus
April 14-17

Tuesday, April 14

Calculus Unit: Applications of Integrals
Lesson 1: Warmup and Quiz

Objective: Be able to do this by the end of this lesson.
1. Practice setting up and solving horizontal area problems.
2. Take a quiz on the week's work!

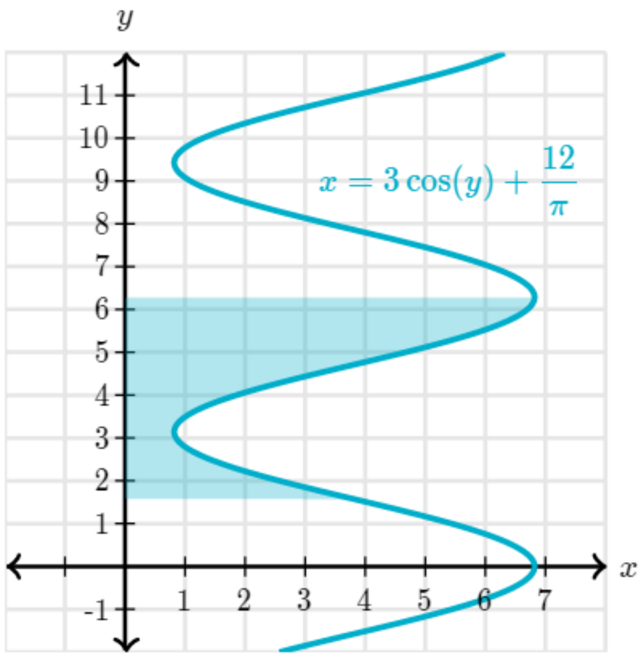
Introduction to Lesson 1
This is a practice day! Warmup by solving some problems introduced last week, then take the quiz. Before you take the quiz, I would recommend spending some time reviewing the packet last week to refresh your memory.

Lesson 1 – Warmup and Quiz

Do this warmup question, check your answer, then flip the page to start the quiz!

1)

The curve $x = 3\cos(y) + \dfrac{12}{\pi}$ is graphed.



Calculus I – Quiz on Finding 2-D Areas Between Curves

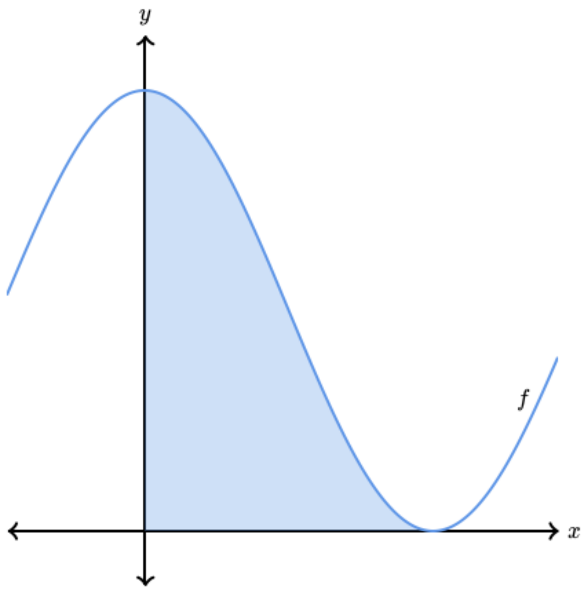
Name: ___________________________________

1)

What is the area of the region between the graphs of $f(x) = x^2 + 12x$ and $g(x) = 3x^2 + 10$ from $x = 1$ to $x = 4$?

2)

The shaded region is bounded by the graph of the function $f(x) = 2 + 2\cos x$ and the coordinate axes.



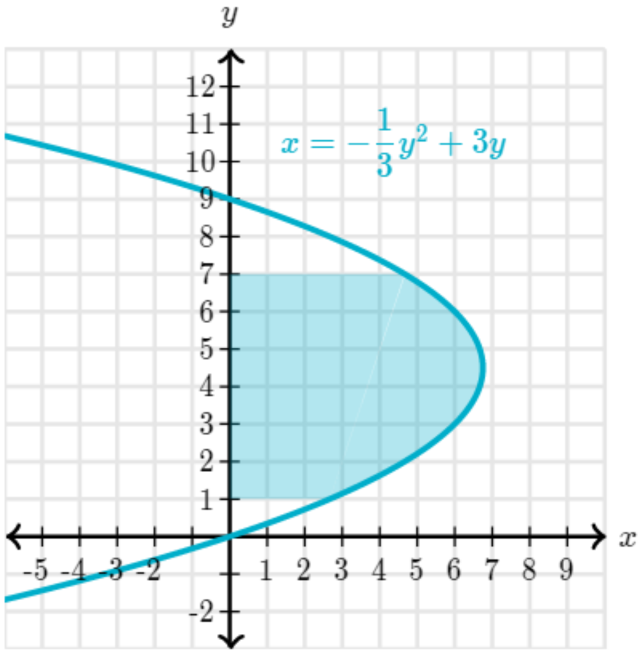
What is its area?

3)

What is the area of the region enclosed by the graphs of $f(x) = \sqrt{x+7}$ and $g(x) = 0.5(x+7)$?

4)

The curve $x = -\frac{1}{3}y^2 + 3y$ is graphed.



What is the area bounded by the curve, the y-axis, the line $y = 1$ and the line $y = 7$?

Bonus:

Let $f(x) = x - 4$ and $g(x) = (x - 4)^3$.

Find the sum of the areas enclosed by the graphs of f and g between $x = 3$ and $x = 5$.

Use a graphing calculator and round your answer to three decimal places.

Quiz

1) $\int_1^4 \left(x^2+12x-(3x^2+10)\right)dx = 18$

2) $\int_0^{\pi} (2+2\cos x)\,dx = 2\pi$

3) $\int_{-7}^{-3} \left(\sqrt{x+7}-(0.5(x+7))\right)dx = \frac{4}{3}$

4) $\int_1^7 \left(-\frac{1}{3}y^2+3y\right)dy = 34$

~~Bonus:~~ $f(x) = \frac{1}{2}x^3 - 2x + 3$

$g(x) = x^2 + \frac{7}{2}x - 3$

$\int_{-3}^{1} (f(x)-g(x))\,dx + \int_1^4 (g(x)-f(x))\,dx$ [illegible]

$= 26.67 + 12.375 \approx 39.042$

Bonus: $f(x) = x-4$

$g(x) = (x-4)^3$

$\int_3^4 (g(x)-f(x))\,dx + \int_4^5 (f(x)-g(x))\,dx = 0.25 + 0.25 = 0.5$

Wednesday, April 15

Calculus Unit: Applications of Integrals
Lesson 2: Finding Volume Using Cross Sections

Objective: Be able to do this by the end of this lesson.
1. Find the volume of solids when cross sections are squares and rectangles.

Introduction to Lesson 2
Now we're going to integrate into the third dimension. Take a look at the first example very carefully. If you're having trouble visualizing the volume of the solid, watch the Khan Academy video under Applications of Integrals titled "Volume with cross sections: squares and rectangles (no graph)." If you're still having trouble, send me an email! Today, think about how adding up an infinite number of rectangles and squares with tiny, tiny widths can give you the volume of a solid. It's sort of like stacking playing cards on top of each other. After you stack enough, you start to get volume that increases the more cards you stack.

Lesson 2 - 1

Volumes with cross sections

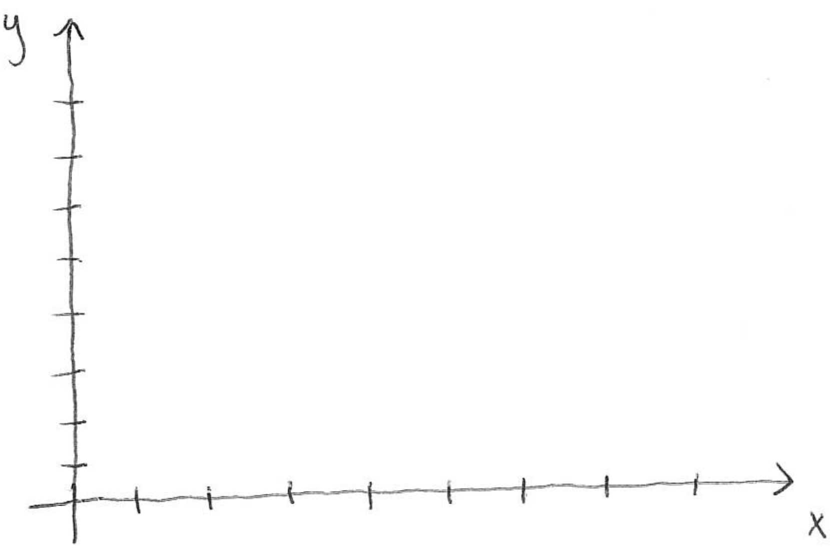
The base of a solid is the region enclosed by $y = -x^2 + 6x - 1$ and $y = 4$.

Cross sections of the solid perpendicular to the x-axis are rectangles whose height is x.

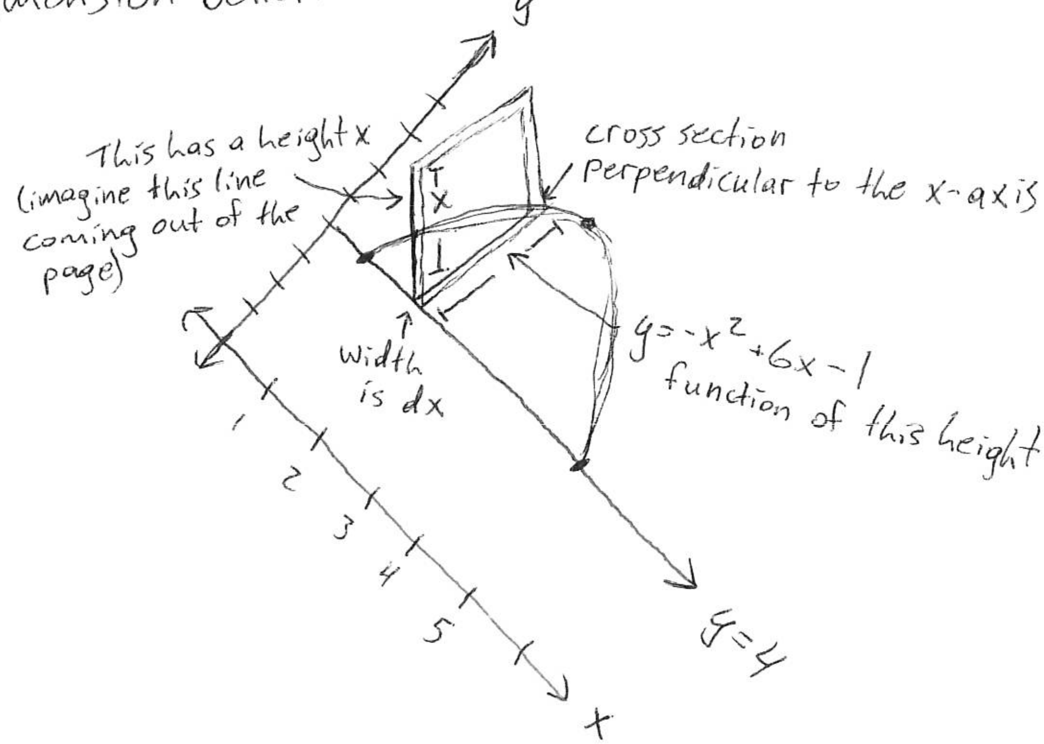
We want to express the volume of this solid with a definite integral. Let's look at how we can do this!

What's the first thing on your mind? Well, hopefully you're wondering about where these functions intersect. Work that out below before we go on.

Now try to sketch a graph of the two functions below:



You should have gotten intersection points at $x=1$ and $x=5$. And the graph should look as follows, though I am going to tilt it so we can see the 3rd dimension better.



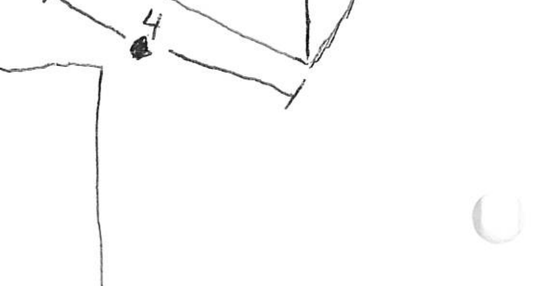
What is the volume of the cross section?

It will be the length, $-x^2+6x-1$, times the width, dx, times the height, x. Putting all that together, we get

$$V_{cross\ section} = x(-x^2+6x-1)\,dx = (-x^3+6x^2-x)\,dx$$

But how do we get the volume of the whole solid?

Remember, we want the whole thing:



Why not integrate?

$$\int_1^5 (-x^3+6x^2-x)\,dx$$ Evaluate below

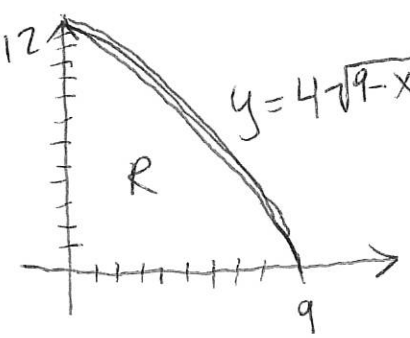
a)

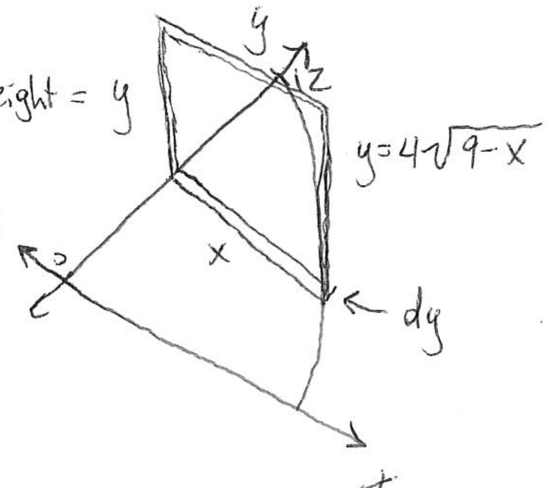
Answer: a) 80 units^3

Lesson 2 - 3

Let's look at another example:

Let R be the region enclosed by $y=4\sqrt{9-x}$ and the x and y-axes in the first quadrant. This region R is the base of a solid. For each y-value, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose base lies in R and whose height is y.

2D: 

3D: 

$$V_{\text{cross section}} = y \cdot x \cdot dy$$

We need to solve $y=4\sqrt{9-x}$ ~~in terms of~~ for x so we can plug it into $V_{\text{cross section}}$ above.

$$y = 4\sqrt{9-x}$$

$$\left(\frac{y}{4}\right)^2 = \sqrt{9-x}^{\,2}$$

$$\frac{y^2}{16} = 9 - x$$

$$x = 9 - \frac{y^2}{16}$$

$$V_{\text{cross section}} = y\left(9 - \frac{y^2}{16}\right) dy$$

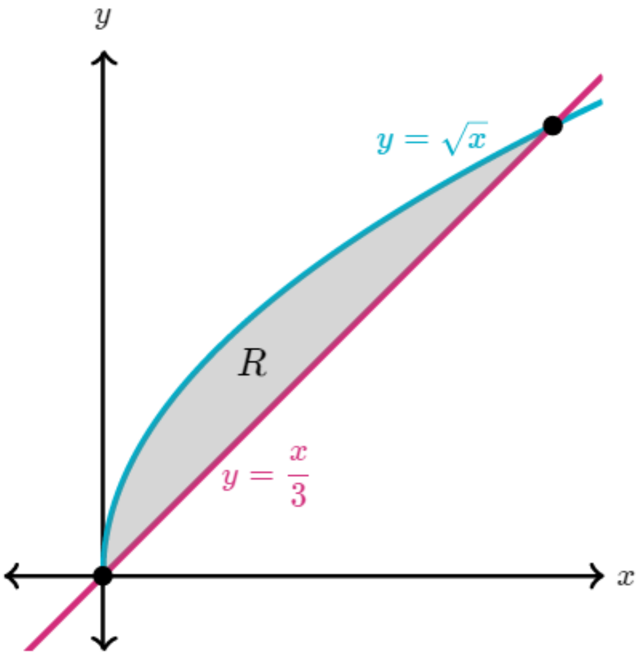
Now set up the integral and evaluate: $\int_0^{12} y\left(9-\frac{y^2}{16}\right) dy$

Try to solve the integral. You should get: 324 units^3

Lesson 2 Exercises – Volumes with Cross Sections

1)

Let R be the region enclosed by the curves $y = \sqrt{x}$ and $y = \dfrac{x}{3}$.

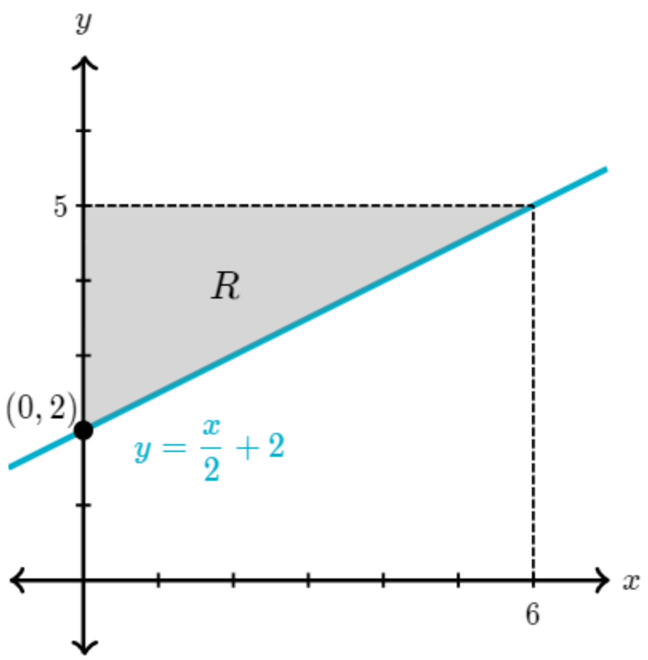


Region R is the base of a solid whose cross sections perpendicular to the x-axis are squares.

Write the definite integral. As a challenge, evaluate to find the volume.

2)

Let R be the region enclosed by $y = \frac{x}{2} + 2$, the line $y = 5$, and the y-axis.

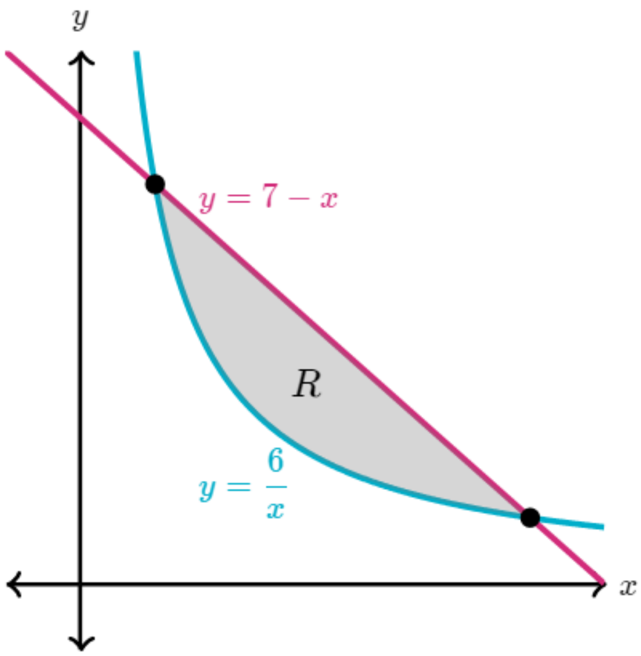


Region R is the base of a solid whose cross sections *perpendicular to the y-axis* are squares.

Write the definite integral then find the volume.

3)

Let R be the region enclosed by the curves $y = \frac{6}{x}$ and $y = 7 - x$.



Region R is the base of a solid whose cross sections perpendicular to the x-axis are squares.

Find the volume.

You are finished for the day. There is a number 4 on the answer key, but there is not an Exercise #4 for you to do. Go outside and play!

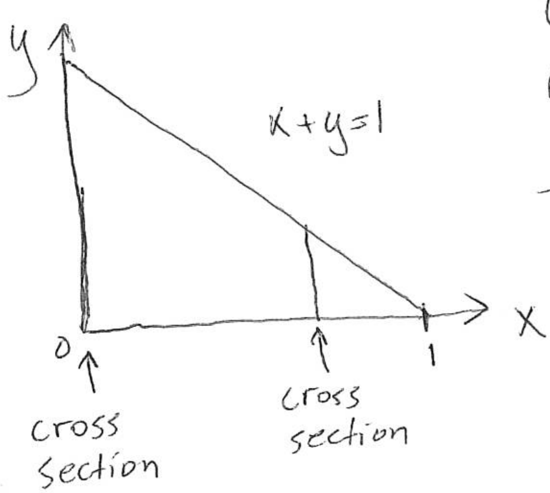
Thursday, April 16

Calculus Unit: Applications of Integrals
Lesson 3: Finding Volume Using Cross Sections

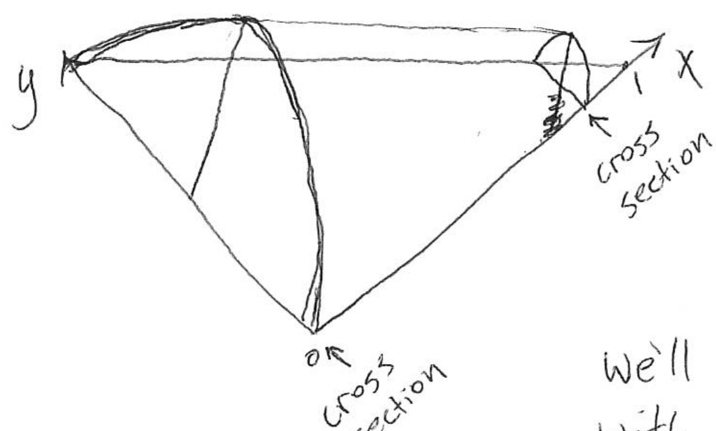
Objective: Be able to do this by the end of this lesson.
1. Find the volume of solids when cross sections are semi-circles.

Introduction to Lesson 3
Today we're going to do the same thing as yesterday, except instead of making squares or rectangle cross-sections, we're going to make semi-circles. Notice though how we're still taking infinitely tiny widths dx or dy and multiplying by the area of the base of the solid to get the volume. If you need extra help, watch the Khan Academy video titled "Volume with cross sections: semi-circle." As always, send me an email if you get stuck somewhere.

Lesson 3-1



Let's call this triangle to the left the base of a 3-D figure. If we take $\perp$ cross sections like we did in the last lesson, we'll get semi-circles.

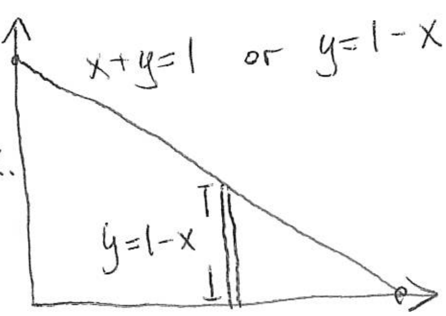


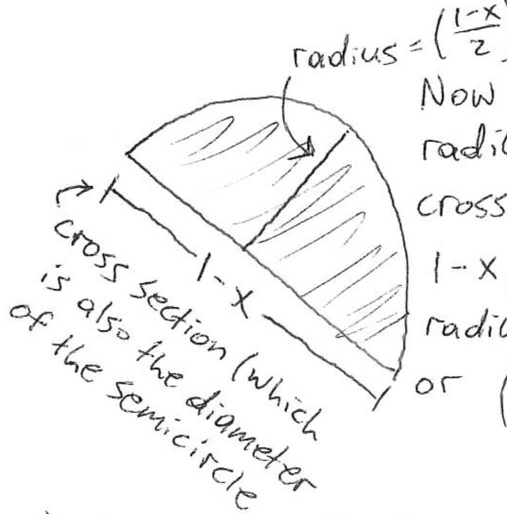
Now we can find the volume.

$$V_{\text{cross section}} = V_{\text{disc}}$$

We'll make the cross sections discs with tiny widths.

The height will be 1-x.

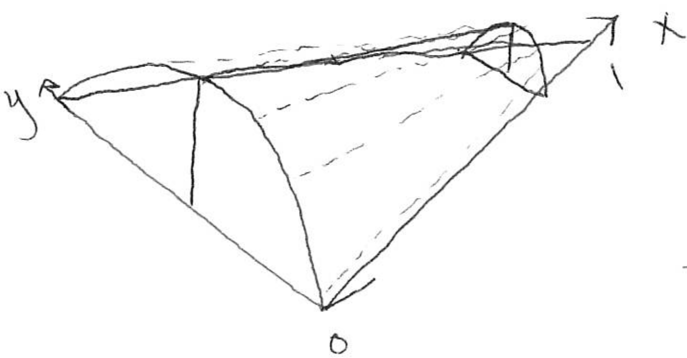


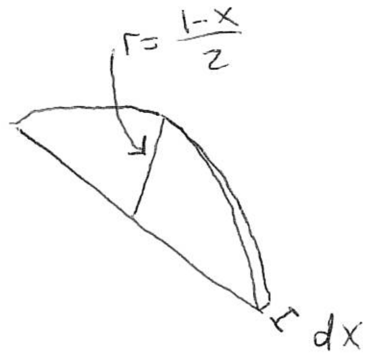


Now let's look at the radius. If the cross section is $1-x$, then the radius will be half, or $\left(\frac{1-x}{2}\right)$

What's the area of the semi-circle?

$$A_{\text{semi-}\circ} = \frac{1}{2}\pi r^2$$
$$= \frac{\pi}{2}\left(\frac{1-x}{2}\right)^2$$





To find the volume, we'll take the area, $A = \frac{\pi}{2}\left(\frac{1-x}{2}\right)^2$ and multiply by the depth, dx.

$$V_{disc} = \frac{\pi}{2}\left(\frac{1-x}{2}\right)^2 dx$$

And to find the volume over the whole figure, we'll expand $\left(\frac{1-x}{2}\right)^2$ and integrate from 0 to 1, the length of the figure on the x-axis.

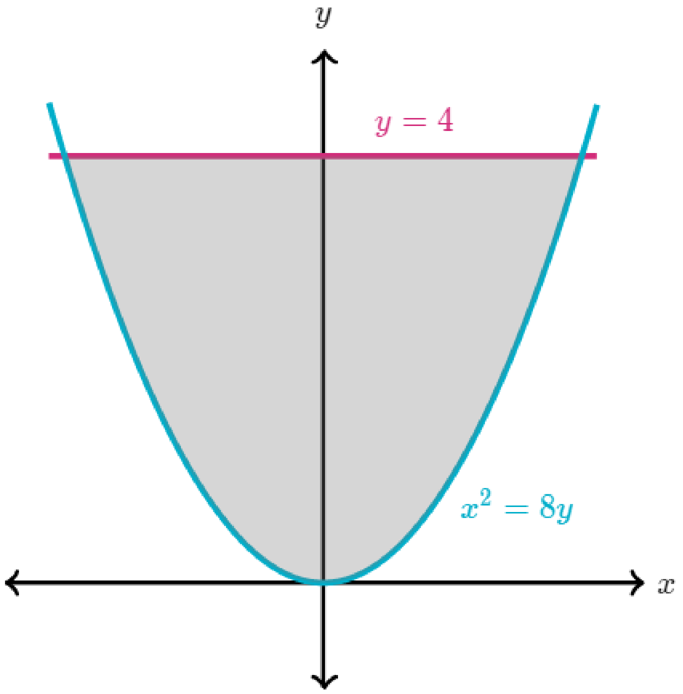
$$V_{figure} = \int_0^1 \frac{\pi}{2} \frac{(x^2 - 2x + 1)\,dx}{4}$$

In the space below, evaluate the integral! You should get $\frac{\pi}{24}$.

Lesson 3 Exercises – Volumes with Cross Sections as Semi-Circles

1)

The base of a solid S is the region bounded by the parabola $x^2 = 8y$ and the line $y = 4$.

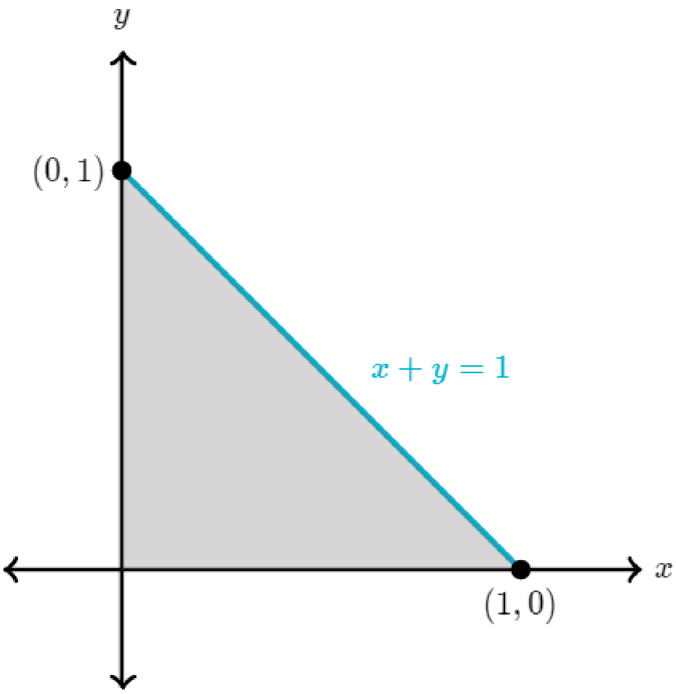


Cross-sections perpendicular to the y-axis are semi-circles.

Find the volume.

2)

The base of a solid S is the triangle enclosed by the line $x + y = 1$, the x-axis, and the y-axis.



Cross-sections perpendicular to the x-axis are semi-circles.

Find the volume.

Friday, April 17

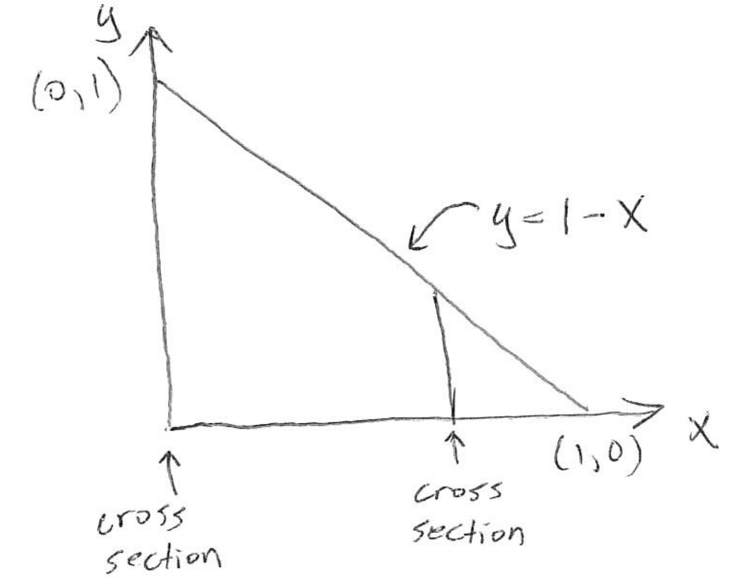
Calculus Unit: Applications of Integrals
Lesson 4: Finding Volume Using Cross Sections

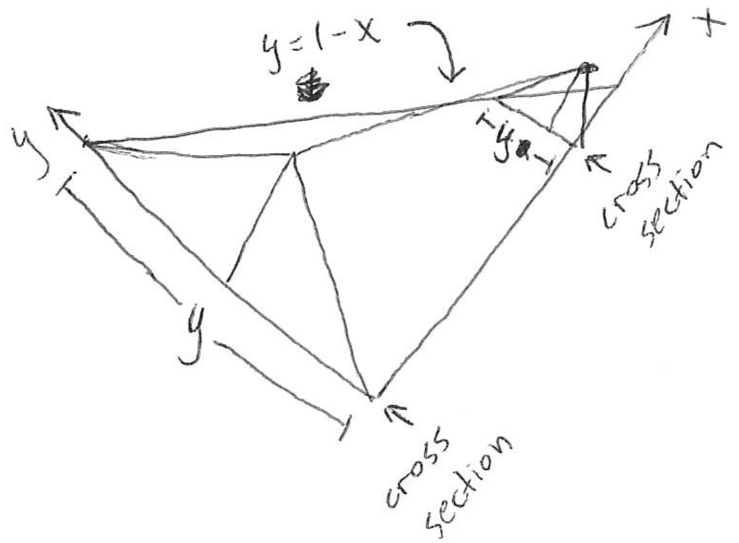
Objective: Be able to do this by the end of this lesson.
1. Find the volume of solids when cross sections are equilateral triangles.

Introduction to Lesson 4
Today we're going to do the same thing as yesterday, except instead of making semi-circle cross sections, we're going to make triangles. Notice though how we're still taking infinitely tiny widths dx or dy and multiplying by the area of the base of the solid to get the volume. If you need extra help today, watch the Khan Academy video titled, "Volume with cross sections: triangle." Send me an email if you need help anywhere along the way and have a great weekend!

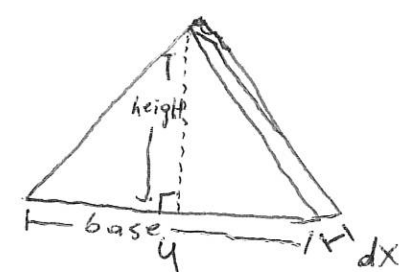
Lesson 4 - 1

Now we're going to do the same thing, except we want the cross sections to be equilateral triangles instead of semi-circles.





If we look at a slice, we see the base of the equilateral triangle is y. Use trig to find the height.



Good, the height will be $\frac{\sqrt{3}}{2}y$

The area will be $A = \frac{1}{2}$ base $\times$ height of the triangle slice

$$A = \frac{1}{2} \cdot y \cdot \frac{\sqrt{3}}{2} y$$

$$A = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} y^2$$

$$A = \frac{\sqrt{3}}{4} y^2$$

Now solve in terms of x: $A = \frac{\sqrt{3}}{4}(1-x)^2$
($y=1-x$)

Finally, write the volume then integral, ~~so~~ then solve.

$$V_{slice} = \frac{\sqrt{3}}{4}(x-1)^2\,dx$$

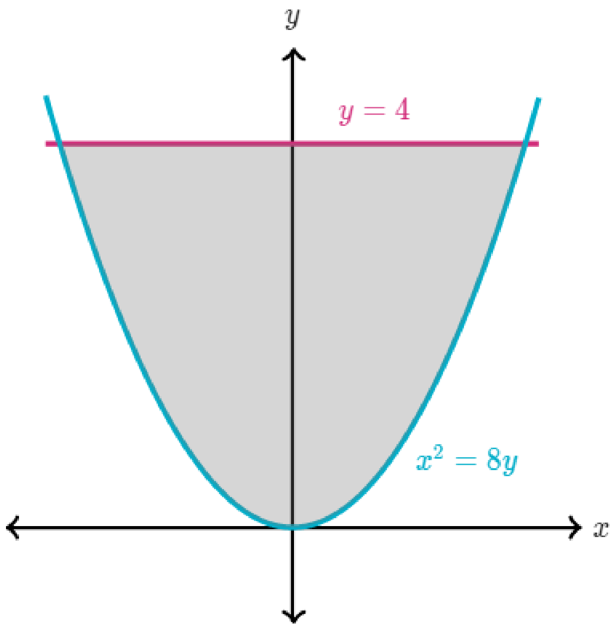
$$V_{solid} = \int_0^1 \frac{\sqrt{3}}{4}(x-1)^2\,dx = \frac{\sqrt{3}}{12}$$

Evaluate the integral in the space below. Make sure you get $\frac{\sqrt{3}}{12}$

Lesson 4 Exercises – Volumes with Cross Sections as Equilateral Triangles

1)

The base of a solid S is the region bounded by the parabola $x^2 = 8y$ and the line $y = 4$.

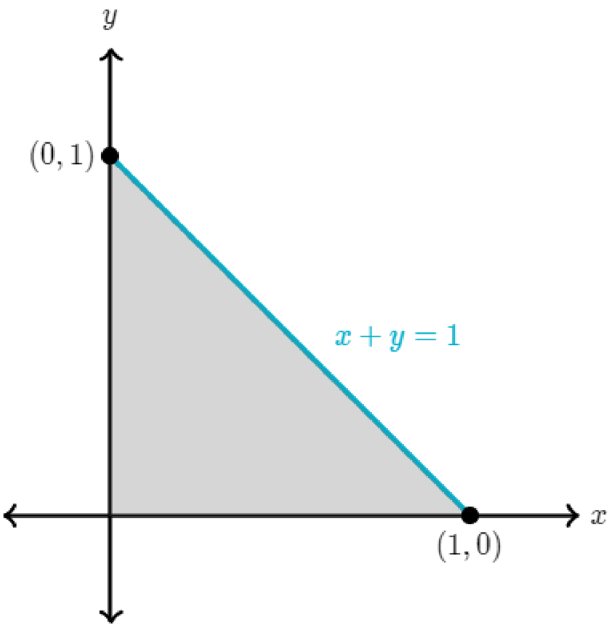


Cross-sections perpendicular to the y-axis are equilateral triangles.

Determine the *exact* volume of solid S.

2)

The base of a solid S is the triangle enclosed by the line $x + y = 1$, the x-axis, and the y-axis.



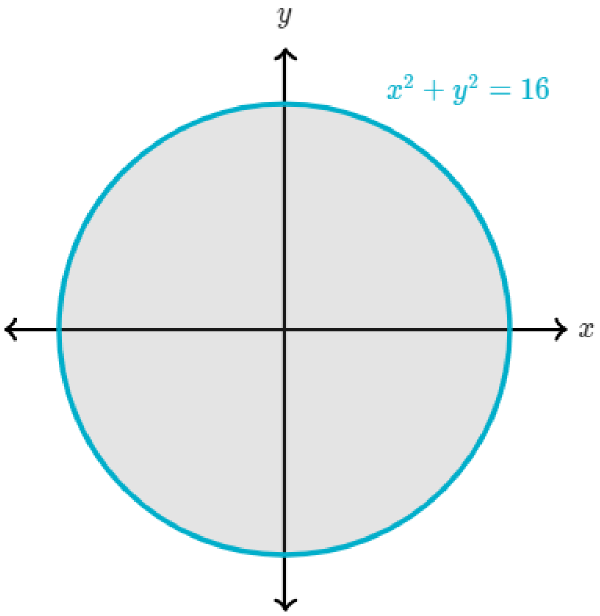
Cross-sections perpendicular to the x-axis are isosceles right triangles with the hypotenuse lying in the base.

Determine the *exact* volume of solid S.

Find the volume.

Challenge:

The base of a solid S is the region bounded by the circle $x^2 + y^2 = 16$.



Cross-sections perpendicular to the x-axis are isosceles right triangles with the hypotenuse lying in the base.

Determine the *exact* volume of solid S.

Answer key
Lesson 2

Lesson 1: 1) $\int_{\frac{\pi}{2}}^{2\pi} \left(3\cos y + \frac{12}{\pi}\right) dy = 15$

1) Intersections: $x = 0$, $x = 9$

$$V_{\text{cross section}} = \text{length} \times \text{width} \times \text{height}$$
$$= \left(\sqrt{x} - \frac{x}{3}\right)(dx)\left(\sqrt{x} - \frac{x}{3}\right)$$

(length and height are equal because problem says the side perpendicular to cross section's base makes a square)

$$= \left(\sqrt{x} - \frac{x}{3}\right)(dx)$$

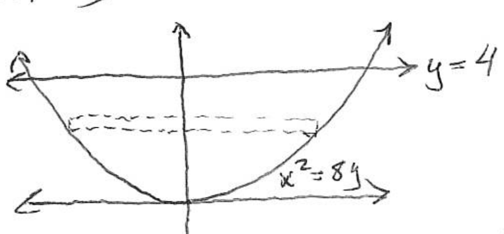
$$\boxed{\int_0^9 \left(\sqrt{x} - \frac{x}{3}\right)^2 dx = \frac{27}{10}}$$

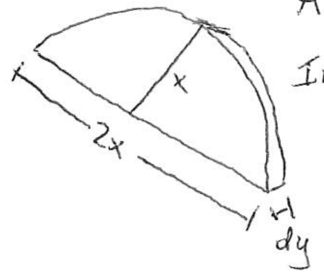
2) $\int_2^5 (2y-4)^2\, dy = 36$

3) $\int_1^6 \left(7 - x - \frac{6}{x}\right)^2$

4) $\int_0^6 \left(4 - \frac{y^2}{9}\right) y \cdot dy = 36$

Lesson 3

1) 



$A = \frac{1}{2}\pi x^2 = \frac{\pi}{2} x^2$

In terms of y:

$A = \frac{\pi}{2}(8y) = 4\pi y$

$V_{\text{disc}} = 4\pi y\, dy$

$V_{\text{solid}} = \int_0^4 4\pi y\, dy = 32\pi$

2) $y = 1 - x$

$A = \frac{1}{2}\pi\left(\frac{y}{2}\right)^2 = \frac{\pi}{8} y^2$

In terms of x...

$A = \frac{\pi}{8}(1-x)^2 = \frac{\pi}{8}(x-1)^2$

$V = \int_0^1 \frac{\pi}{8}(x-1)^2\, dx$

$V = \frac{\pi}{24}$

Challenge: $V_{\text{figure}} = \frac{128\pi}{3}$

Lesson 4

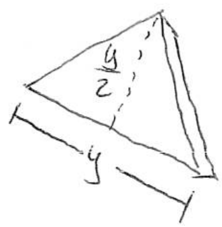
1) $A = \frac{1}{2} \cdot 2x \cdot \sqrt{3}x = \sqrt{3}x^2$

In terms of y: $A = \sqrt{3}(8y)$
(because we're given $x^2 = 8y$)

$$\int_0^4 \sqrt{3}(8y)\,dy = 8\sqrt{3}\int_0^4 y\,dy = \boxed{64\sqrt{3}}$$

2) $y = 1 - x$
$A = \frac{1}{2} \cdot y \cdot \frac{y}{2} = \frac{1}{4}y^2$ $\quad V = \int_0^1 \frac{1}{4}(x-1)^2$

$\boxed{V = \frac{1}{12}}$



In terms of x:
$A = \frac{1}{4}(1-x)^2$
$= \frac{1}{4}(x-1)^2$

Challenge: $\boxed{\frac{256}{3}}$