GreatHearts®

Calculus I

April 20 - April 23

Time Allotment: 40 minutes per day

Student Name: ___________________________________

Teacher Name: ___________________________________

Calculus
April 20-23

GreatHearts®

Packet Overview

Date	Objective(s)	Page Number
Monday, April 20	Disc Method: Rotating Single Functions About X and Y-Axes	2
Tuesday, April 21	Rotating About Other Axes	3
Wednesday, April 22	1. Finding Volume Using Rectangle and Square Cross Sections	4
Thursday, April 23	Quiz	5
Friday, April 24	Off	

Additional Notes: Dear Students: this week we're going to use integrals to enter into the third dimension and find the volumes of solids! Get excited!

Though not required to complete these assignments, Khan Academy's AP Calculus AB series of videos are a helpful resource for supplemental learning. We'll be going over the chapter on Applications of Integrals. If you're having trouble seeing the 3-D pictures on my handwritten notes, they follow closely the videos titled "Volumes with cross sections" in that Applications of Integrals section. Take a look at those while you work through the examples.

Academic Honesty

I certify that I completed this assignment independently in accordance with the GHNO Academy Honor Code.

Student signature:

____

I certify that my student completed this assignment independently in accordance with the GHNO Academy Honor Code.

Parent signature:

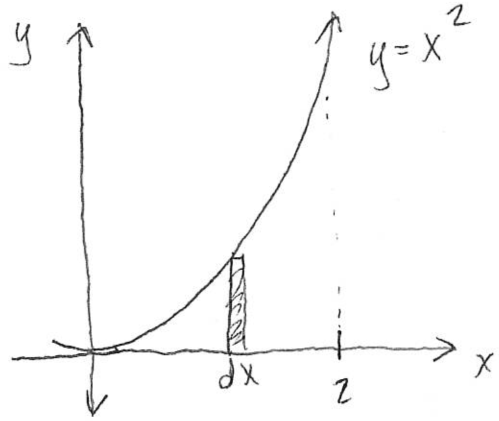
____

Monday, April 20

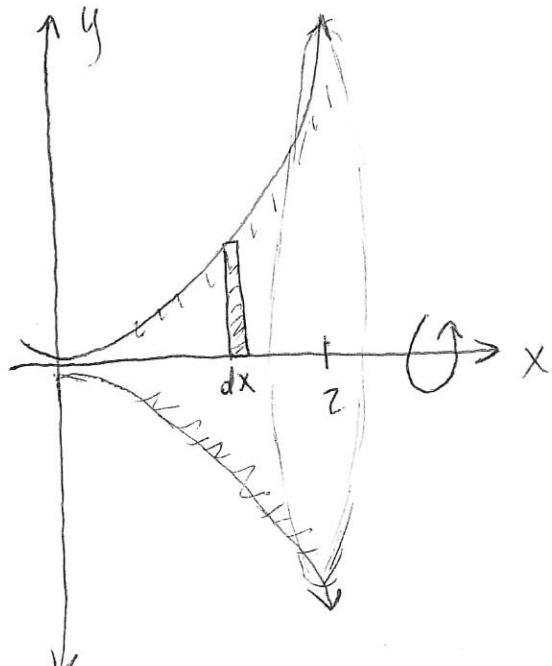
Calculus Unit: Applications of Integrals
Lesson 1: Rotations About X and Y-Axes

Objective: Be able to do this by the end of this lesson.
1. Explain what the disc method is and how it is used for finding cross sections of rotated solids.
2. Rotate a single function about the x-axis and find the volume of the resulting solid.
3. Rotate a single function about the y-axis and find the volume of the resulting solid.

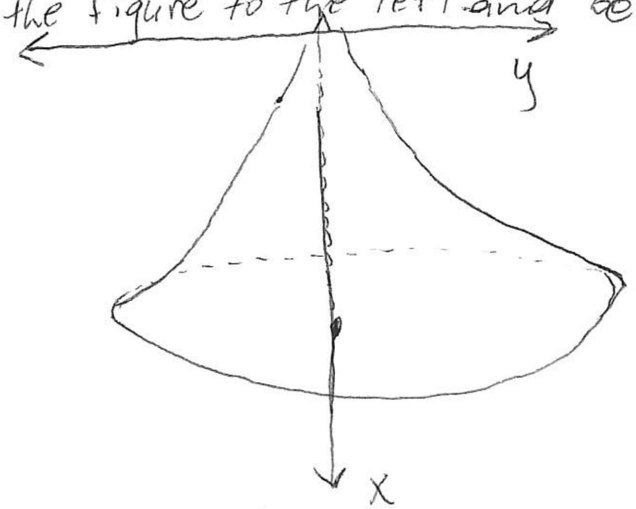
Introduction to Lesson 1
Today we're going to take a single function and rotate it 360° about an axis. As we rotate it, we're looking at all the three dimensional points it sweeps out. Imagine taking a long-exposure photograph of a glow stick at night as you wave it sideways. The resulting picture will show all the points in space where the glow stick has swept. To find the volume when we rotate a function about one of the axes, we again find the cross section of the solid. The shape of these cross sections will be discs, and so our method of finding the volume of one of these discs then integrating over the entire solid will be called the disc method of integration.



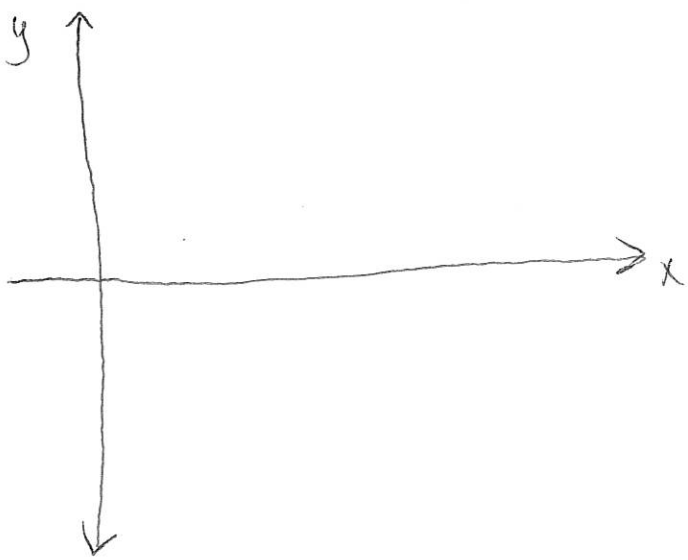
$\int_0^2 x^2\,dx$ We already know that if we evaluate this integral, we'll get the area under the curve of $y=x^2$ from 0 to 2. But what if we want to take the curve on the same interval and rotate it about the x-axis?



That rotation gives you a volume that looks something like the figure to the left and below:

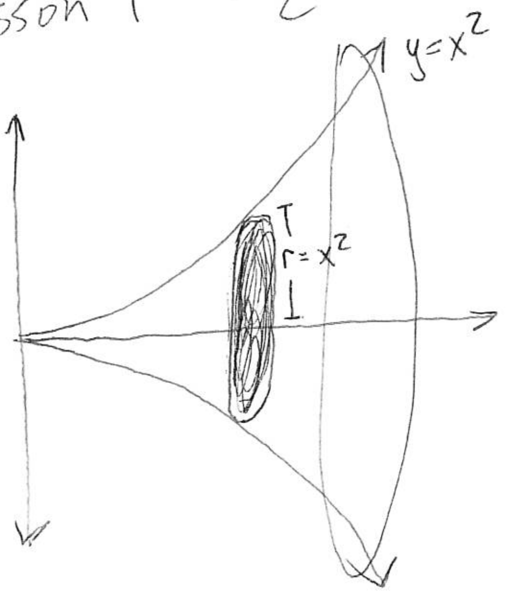


To find the volume, what if we rotated each rectangle of dx width about the x-axis?

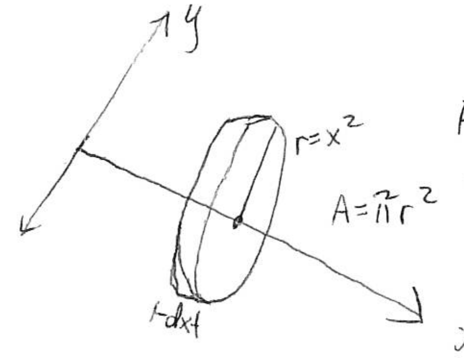


On the graph to the left, sketch what one rotated rectangle might look like. What shape does it make?

Lesson 1 - 2



Hopefully, you got a shape that looks like a disc.



From another angle:

To find the volume, first let's find the area then the volume of one disc.

$$A_{\text{disc face}} = \pi r^2$$
$$= \pi (x^2)^2$$

$$V_{disc} = (A_{\text{disc face}})(dx)$$ Remember, the depth of any disc is dx thick.

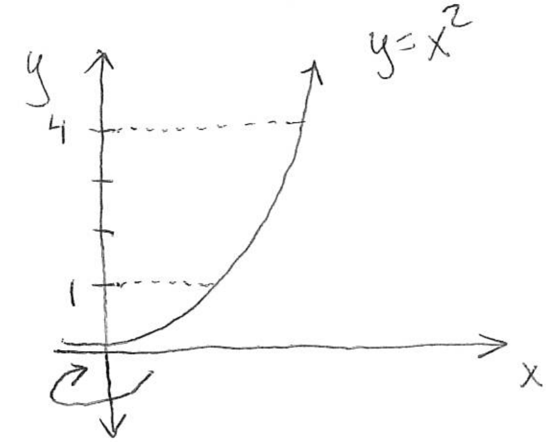
$$V_{disc} = \pi x^4 \, dx$$

Finally, how do we find the volume of the whole solid? The same way we did last week:

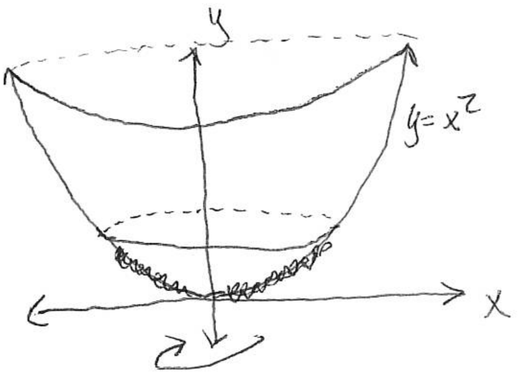
$$V_{solid} = \int_0^2 \pi x^4 \, dx$$ ← Work this out in the space below

Answer: $\frac{32\pi}{5}$

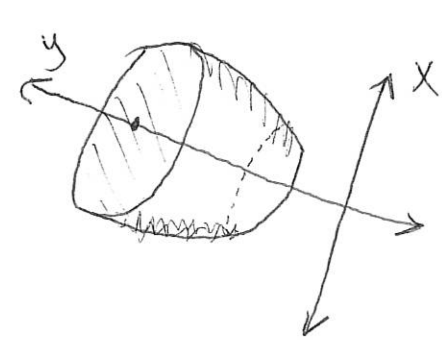
Lesson 1 - 3



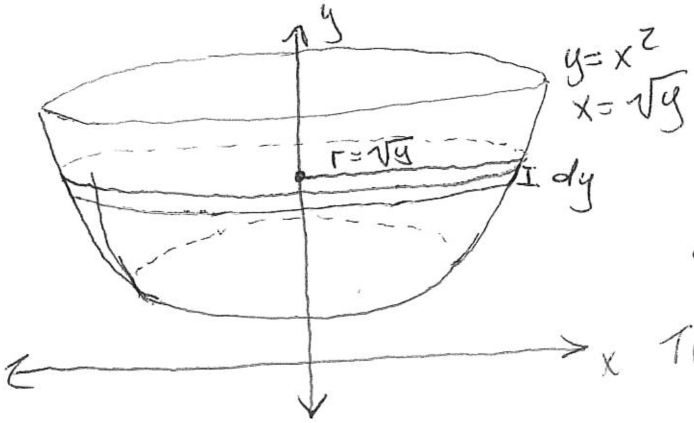
Now let's take the same function and instead of rotating about the x-axis, we'll rotate about the y-axis. In addition, we'll make our boundaries $y=1$ and $y=4$.



After the rotation, we'll get a 3-D solid that will look like this.



Think about how we might divide up this solid into discs.



We want to stack the discs, so we'll give them dy thicknesses. The radius of the discs will be the distance from the y-axis to a point on the function, or $x = \sqrt{y}$.

The area of the discs will then be

$$A_{disc} = \pi r^2$$
$$= \pi (\sqrt{y})^2$$
$$= \pi y$$

Finally, the thickness is dy.

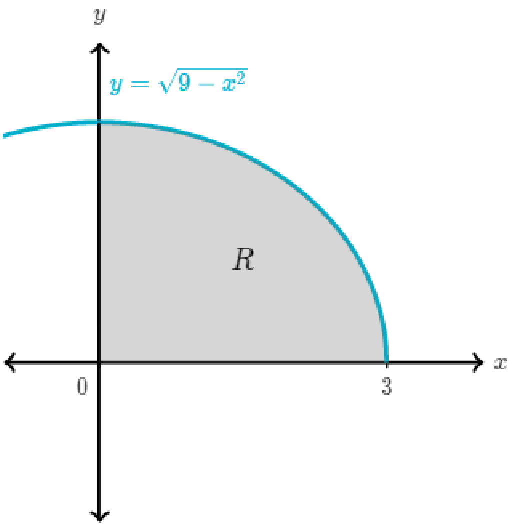
$$V_{disc} = \pi y \, dy$$

And the volume of the solid: $\int_1^4 \pi y \, dy = \frac{15\pi}{2}$

Calculus I – Disc Method Exercises

1)

Let R be the region enclosed by the positive x-axis, the positive y-axis, and the curve $y = \sqrt{9 - x^2}$.

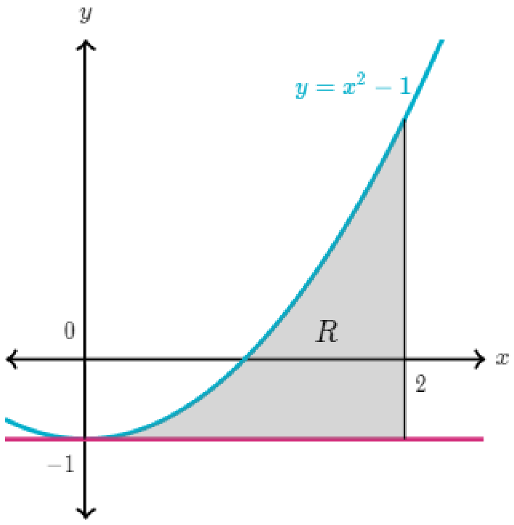


A solid is generated by rotating R about the x-axis.

What is the volume of the solid?
Give an exact answer in terms of π.

2)

Let R be the region enclosed by the line $y = -1$, the line $x = 2$, and the curve $y = x^2 - 1$.

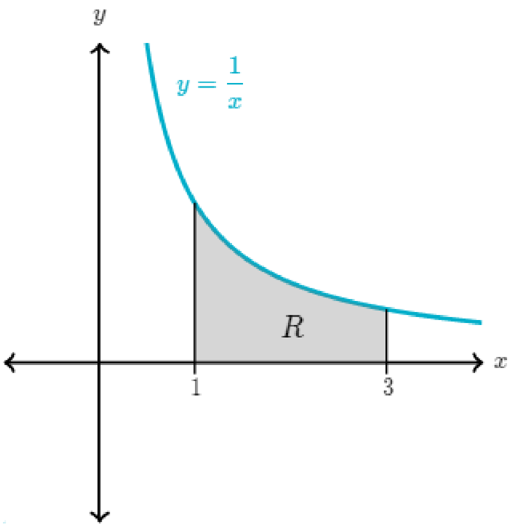


A solid is generated by rotating R about the line $y = -1$.

Find the exact volume of the solid.

3)

Let R be the region enclosed by the x-axis, the line $x = 1$, the line $x = 3$, and the curve $y = \dfrac{1}{x}$.



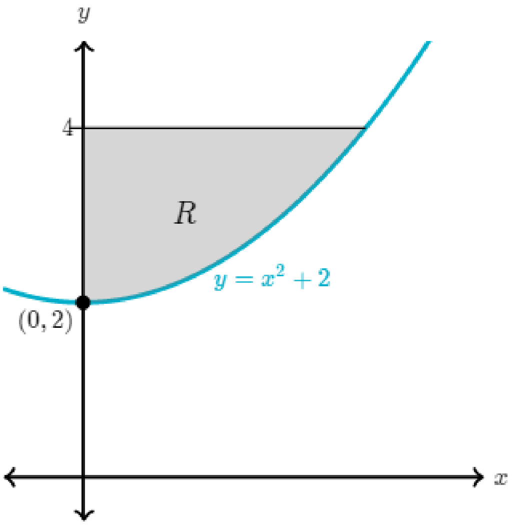
A solid is generated by rotating R about the x-axis.

What is the volume of the solid?

Give an exact answer in terms of π.

4)

Let R be the region in the first quadrant enclosed by the y-axis, the line $y = 4$, and the curve $y = x^2 + 2$.



A solid is generated by rotating R about the y-axis.

What is the volume of the solid?

Give an exact answer in terms of π.

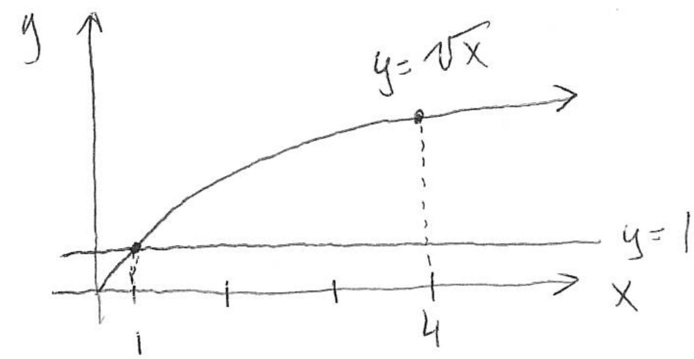
Tuesday, April 21

Calculus Unit: Applications of Integrals
Lesson 2: Rotating about lines that are not the x or y-axis.

Objective: Be able to do this by the end of this lesson.
1. Find volume using disc method while account for the axis of rotation .

Introduction to Lesson 2
Today we'll use the same disc method to find the volume of a solid created by rotating a function about an axis that is not the x or y-axis. Think rotating about the line $y = 1$, or $x = 5$. The biggest difference to pay attention to is the value for the radius of your disc. Think about adding or subtracting quantities to get the correct radius length.

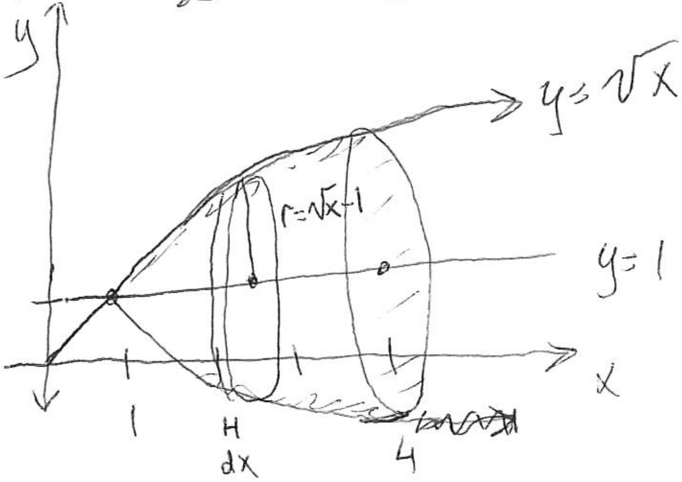
Lesson 2 - 1



what if instead of rotating about the y-axis, we rotate about a horizontal line, say $y=1$, and want to find the volume of a solid from $x=1$ to $x=4$

Go ahead and give it a shot below. It's OK to be wrong. But try before you turn the page. You might surprise yourself!

Lesson 2 - 2



We need to find the area of a disc on the inside:

$$A_{disc} = \pi r^2$$

What is r?

It's $\sqrt{x} - 1$ because we want to go up to the function $\sqrt{x}$ from the x-axis but then make it 1 unit shorter so it only goes from $y=1$ to $y=\sqrt{x}$.

$$A_{disc} = \pi\left(\sqrt{x} - 1\right)^2$$

And the thickness is dx, so

$$V_{disc} = \pi\left(\sqrt{x} - 1\right)^2 dx$$

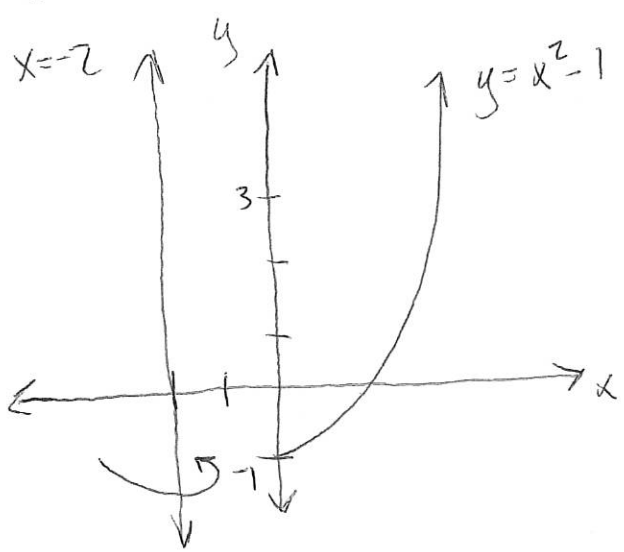
Finally, the volume of the solid:

$$V_{solid} = \int_1^4 \pi\left(\sqrt{x} - 1\right)^2 dx$$

Try to evaluate this integral below:

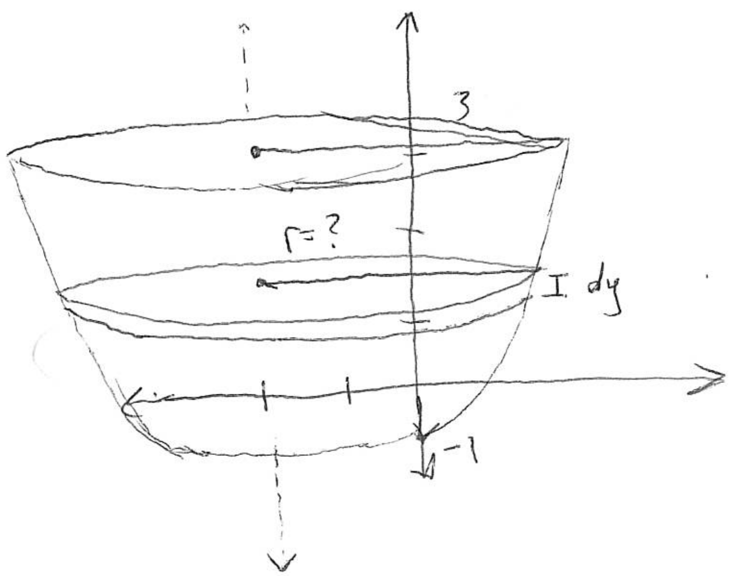
Answer: $V_{solid} = \frac{7\pi}{6}$

Lesson 2 - 3



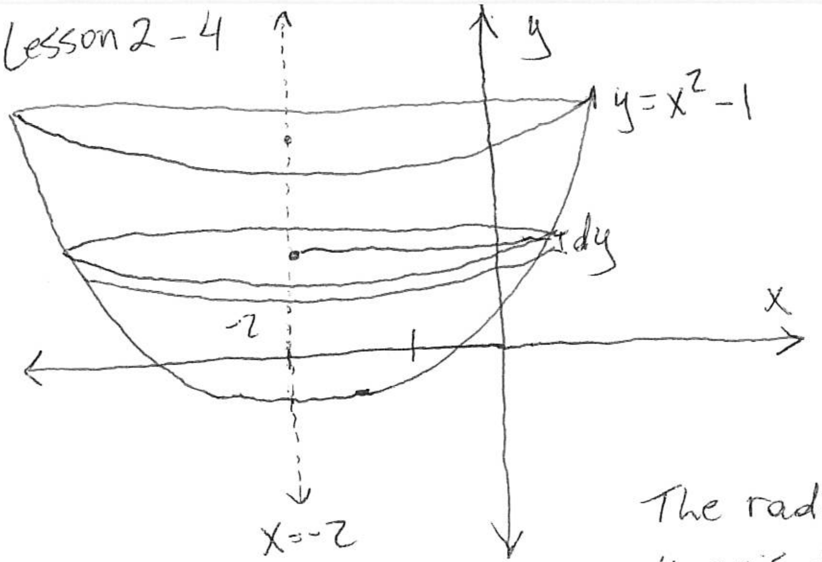
Now we've got a function that we'd like to rotate about the axis $x=-2$.

What will that look like?



See if you can set this up below. Try to find the volume of the solid. Integrate from $y=-1$ to $y=3$

Lesson 2 - 4



First let's find the radius, and let's find it in terms of y:

$$y = x^2 - 1$$

$$x^2 = y + 1$$

$$x = \sqrt{y+1}$$

The radius will be the distance from the y-axis to the function ($\sqrt{y+1}$) plus 2 to get to the center of the circle at $x=-2$.

$$r(y) = \sqrt{y+1} + 2$$

Then $A(y)_{disc} = \pi\left(\sqrt{y+1}+2\right)^2$

And $V_{solid} = \int_{-1}^{3} \pi\left(\sqrt{y+1}+2\right)^2 dy$

Practice your integration skills and evaluate below! u-substitution can be very helpful here.

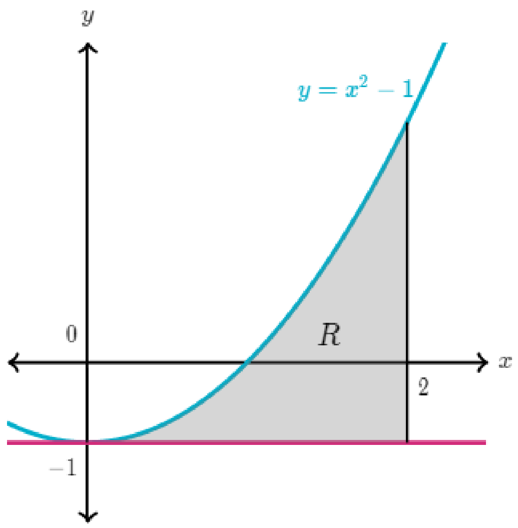
Answer: $V = \pi \cdot \frac{y^2}{2} + \frac{8}{3}(y+1)^{3/2} + 5y \Big|_{-1}^{3} = \frac{272\pi}{6} = \frac{136\pi}{3}$

Lesson 2 Exercises

Calculus I – Disc Method: Rotating About Other Axes

1)

Let R be the region enclosed by the line $y = -1$, the line $x = 2$, and the curve $y = x^2 - 1$.

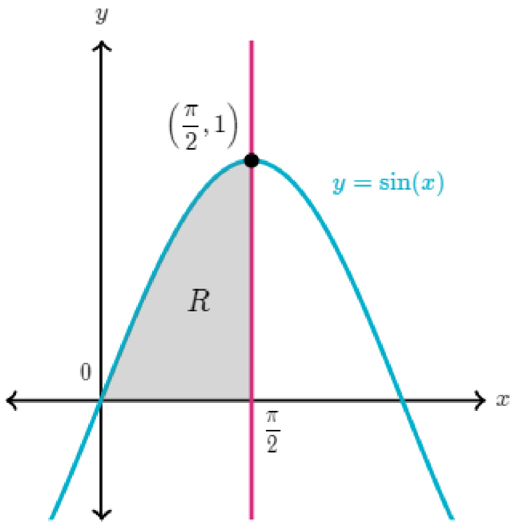


A solid is generated by rotating R about the line $y = -1$.

Find the exact volume of the solid.

2)

Let R be the region to the left of the line $x = \frac{\pi}{2}$ and enclosed by that line, the x-axis, and the curve $y = \sin(x)$.

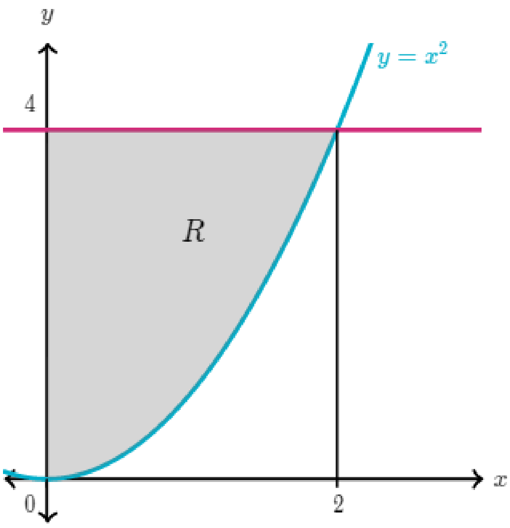


A solid is generated by rotating R about the line $x = \frac{\pi}{2}$.

Set up the definite integral to find the volume of the solid. You don't have to evaluate.

3)

Let R be the region enclosed by the y-axis, the line $y = 4$ and the curve $y = x^2$.



A solid is generated by rotating R about the line $y = 4$.

What is the volume of the solid?

Give an exact answer in terms of π.

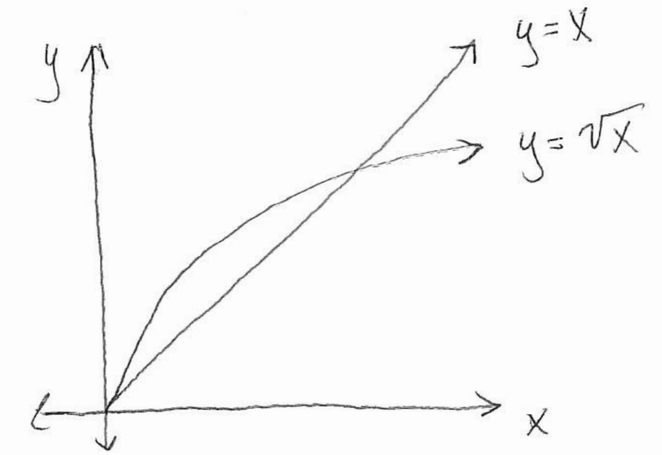
Wednesday, April 22

Calculus Unit: Applications of Integrals
Lesson 3: Finding Volume of Multiple Functions Rotated About an Axis

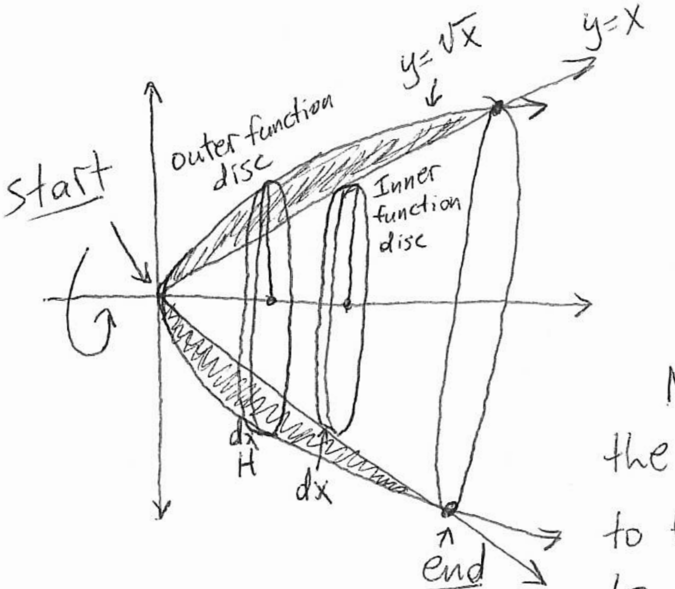
Objective: Be able to do this by the end of this lesson.
1. Draw three dimensional rotations of multiple functions rotated about an axis.
2. Find the inner radius and outer radius of resulting rotation.
3. Draw a washer to visualize inner and outer radii.
4. Integrate over origin to intersection point of functions.

Introduction to Lesson 3
Today we're going to find the area between two functions and then rotate that area about an axis. When we take cross sections of the result, they end up looking like hollowed out discs, or washers. We take the volume of one cross sectional washer and integrate over the boundaries of the solid in the same way we did with the disc method. Because of the washer shape, we call this method the washer method.

Lesson 3 - 1



What if we have two functions and want to rotate the area between them about the x-axis?



What's interesting is that when we rotate it, we get a similar shape as just the $y = \sqrt{x}$ function, except there's a cone shape carved out of it.

Now let's use the disc Method to find the volume. Except this time, we want to find hollow discs, then add them together to find the volume. To make the hollow discs, we need to take the outer function and subtract off the inner function. Before we do that, though, we need to find the limits of integration. We want to start integrating at the origin and end integrating at the intersection of $y = x$ and $y = \sqrt{x}$. Let's find that intersection by setting the two functions equal to one another:

$$x = \sqrt{x}$$
$$x^2 = \sqrt{x}^2$$
$$x^2 = x$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0 \text{ or } 1$$

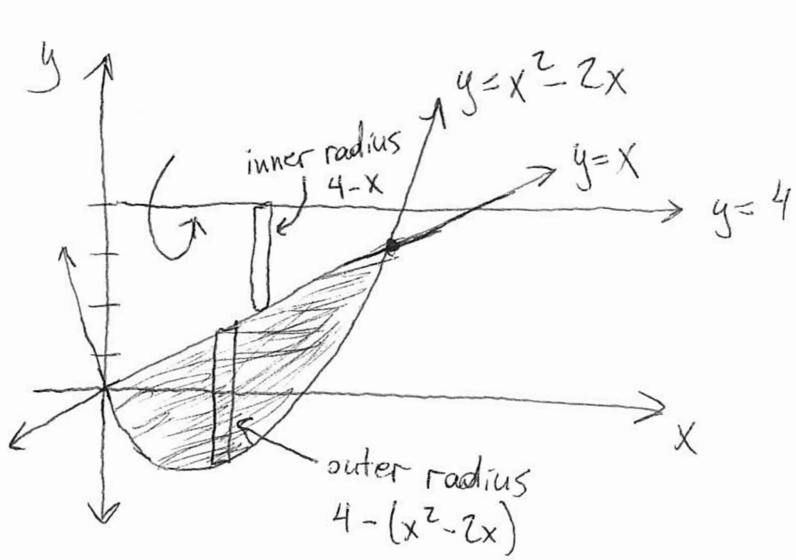
Now we know we're integrating from 0 to 1. So we just want to subtract the two volumes to get the volume of the solid:

$$V_{solid} = \underbrace{\int_0^1 \pi(\sqrt{x})^2\,dx}_{\text{outer function}} - \underbrace{\int_0^1 \pi(x)^2\,dx}_{\text{inner function}}$$

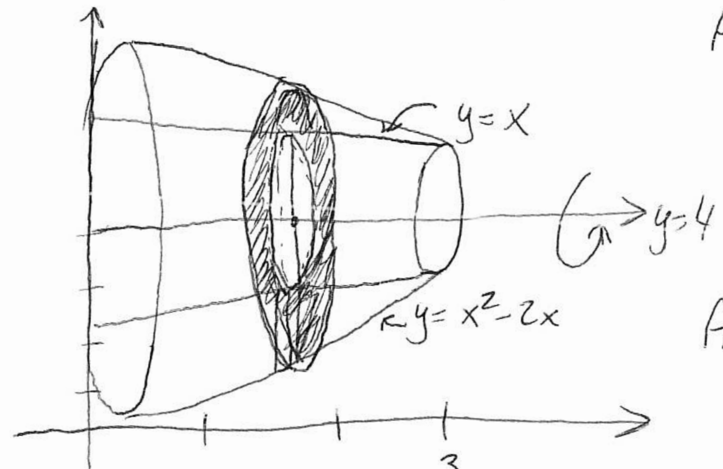
Evaluate below then check your answer.

Answer: $\frac{\pi}{6}$

Lesson 3 - 2



Here we want to rotate the shaded region about the $y=4$ line.



Here's a sketch of what that rotated graph looks like. Remember, the center is hollowed out, looking like a washer.

$$A_{washer} = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$

$$A_{washer} = \pi\left(4-(x^2-2x)\right)^2 - \pi(4-x)^2$$

$$A_{washer} = \pi\left((4-x^2+2x)^2-(4-x)^2\right)$$

Before we integrate and find the volume, we have to know our boundaries. We're starting at $x=0$ and going to the point where $y=x$ and $y=x^2-2x$ intersect. Let's set the two functions equal to one another.

$$x = x^2-2x$$

$$0 = x^2-3x$$

$$0 = x(x-3)$$

$$x=0 \text{ or } x=3$$

e want to integrate from
to 3

Now we have our volume:

$$\int_0^3 \pi\left((4-x^2+2x)^2-(4-x)^2\right)dx$$

As a challenge, try to evaluate the integral.

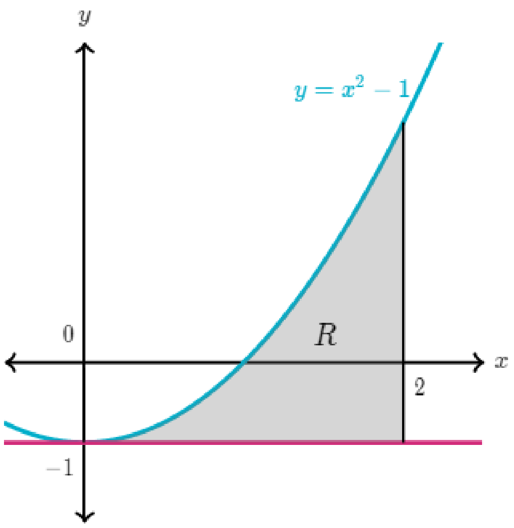
swer: $\frac{153\pi}{5}$

Lesson 3 Exercises

Calculus I – Washer Method

1)

Let R be the region enclosed by the line $y = -1$, the line $x = 2$, and the curve $y = x^2 - 1$.

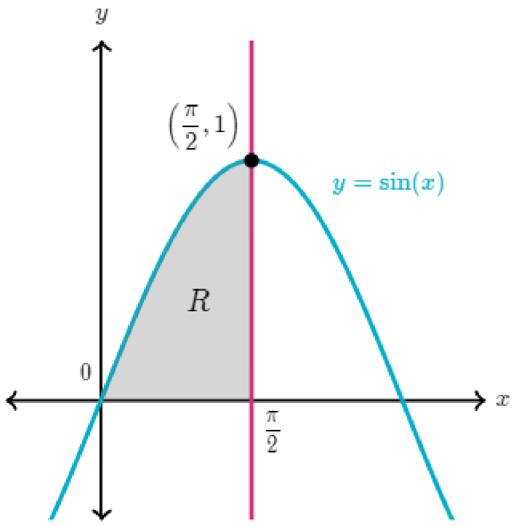


A solid is generated by rotating R about the line $y = -1$.

Find the exact volume of the solid.

2)

Let R be the region to the left of the line $x = \frac{\pi}{2}$ and enclosed by that line, the x-axis, and the curve $y = \sin(x)$.

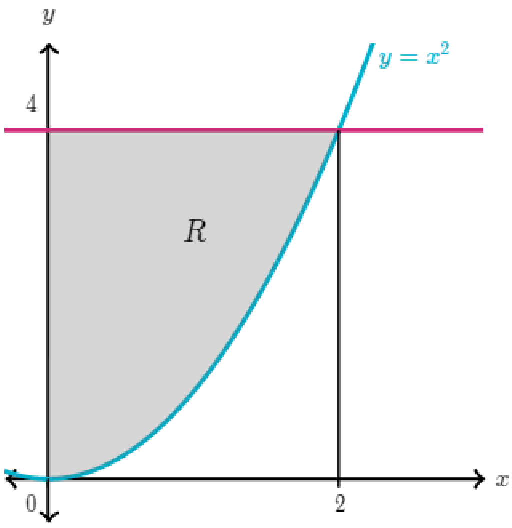


A solid is generated by rotating R about the line $x = \frac{\pi}{2}$.

Set up the definite integral to find the volume of the solid. You don't have to evaluate.

3)

Let R be the region enclosed by the y-axis, the line $y = 4$ and the curve $y = x^2$.



A solid is generated by rotating R about the line $y = 4$.

What is the volume of the solid?

Give an exact answer in terms of π.

Thursday, April 23

Calculus Unit: Applications of Integrals
Lesson 4: Quiz on Disc and Washer Methods

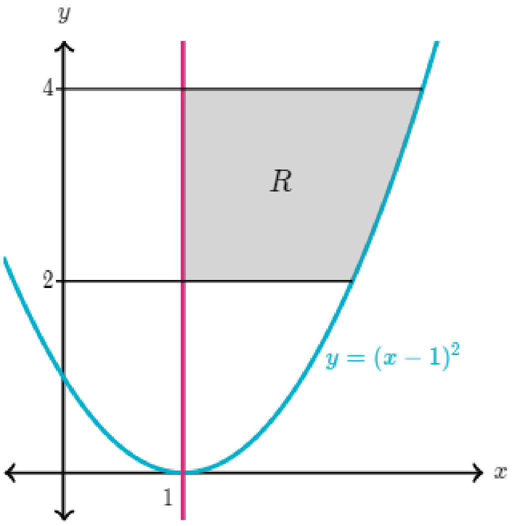
Objective: Be able to do this by the end of this lesson.
1. Demonstrate mastery of disc and washer methods of integration.

Introduction to Lesson 4
The last item for this week is to take a quiz over what we practiced. Take a few minutes to review, then when you're ready, turn the page to begin the quiz.

Calculus I – Quiz on Disc and Washer Methods of Integration

1)

Let R be the region enclosed by the line $x = 1$, the line $y = 2$, the line $y = 4$, and the curve $y = (x - 1)^2$.

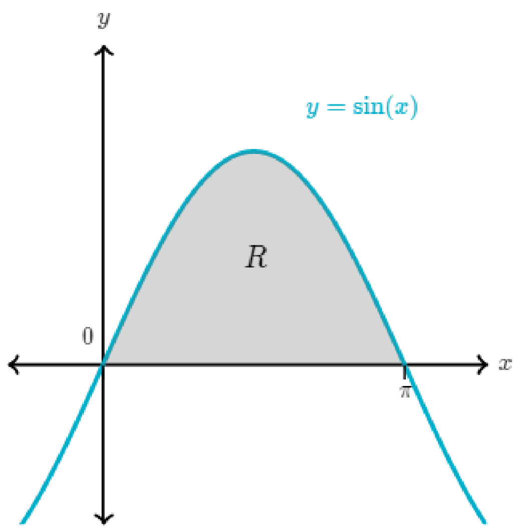


A solid is generated by rotating R about the line $x = 1$.

What is the volume of the solid?
Give an exact answer in terms of π.

2)

Let R be the region enclosed by the x-axis, the y-axis, the line $x = \pi$ and the curve $y = \sin(x)$.

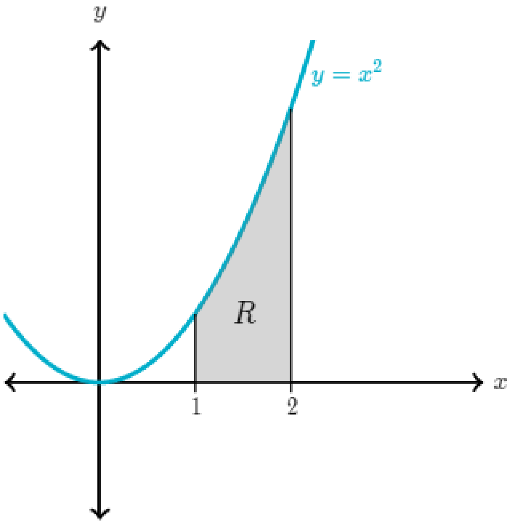


A solid is generated by rotating R about the x-axis.

What is the volume of the solid?

3)

Let R be the region enclosed by the x-axis, the line $x = 1$, the line $x = 2$, and the curve $y = x^2$.

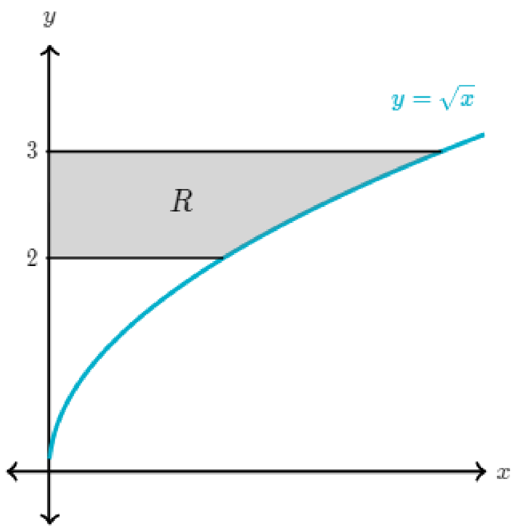


A solid is generated by rotating R about the x-axis.

What is the volume of the solid?
Give an exact answer in terms of π.

4)

Let R be the region enclosed by the y-axis, the line $y = 2$, the line $y = 3$, and the curve $y = \sqrt{x}$.



A solid is generated by rotating R about the y-axis.

What is the volume of the solid?

Give an exact answer in terms of π.

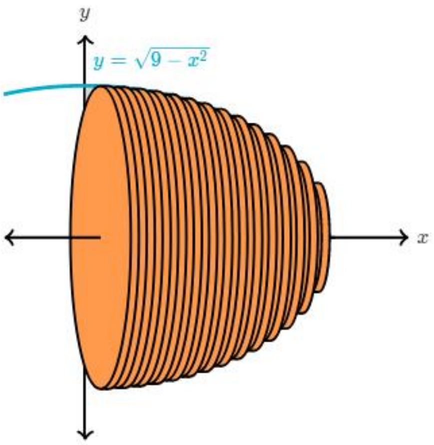
Answer Key

Lesson 1

1)

1 / 5

Let's imagine the solid is made out of many thin slices.



Each slice is a *cylinder*. Let the thickness of each slice be dx and let the radius of the base, as a function of x, be $r(x)$.

Then, the volume of each slice is $\pi[r(x)]^2\,dx$, and we can sum the volumes of infinitely many such slices with an infinitely small thickness using a definite integral:

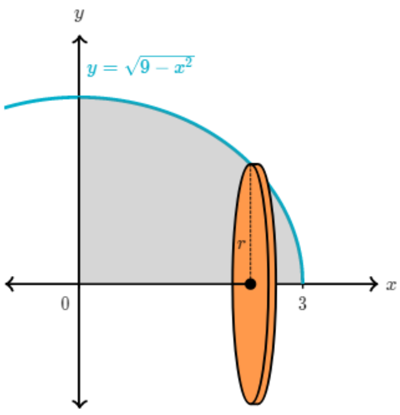
$$\int_a^b \pi[r(x)]^2\,dx$$

This is called the *disc method*.

What we now need is to figure out the expression of $r(x)$ and the interval of integration.

2 / 5

Let's consider one such slice.



The radius is equal to the distance between the curve $y=\sqrt{9-x^2}$ and the x-axis. In other words, for any x-value, $r(x)=\sqrt{9-x^2}$.

Now we can find an expression for the area of the cylinder's base:

$$\pi[r(x)]^2$$

$$=\pi\left(\sqrt{9-x^2}\right)^2$$

$= \pi(9 - x^2)$

3 / 5 The leftmost endpoint of R is at $x = 0$ and the rightmost endpoint is at $x = 3$. So the interval of integration is $[0, 3]$.

Now we can express the definite integral in its entirety!

$$\int_0^3 \left[\pi\left(9 - x^2\right)\right] dx$$

$$= \pi \int_0^3 \left(9 - x^2\right) \, dx$$

4 / 5 Let's evaluate the integral.

$$\pi \int_0^3 \left(9 - x^2\right) \, dx = 18\pi$$

[Show calculation.]

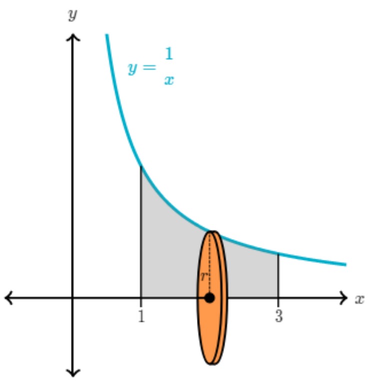
5 / 5 **In conclusion, the volume of the solid is 18π.**

2)

$$\int_0^2 \left(\pi x^4\right) \, dx \quad = 32\pi/5$$

3)

2 / 5 Let's consider one such slice.



The radius is equal to the distance between the curve $y = \frac{1}{x}$ and the x-axis. In other words, for any x-value, $r(x) = \frac{1}{x}$.

Now we can find an expression for the area of the cylinder's base:

$$\pi[r(x)]^2$$

$$= \pi\left(\frac{1}{x}\right)^2$$

$$= \frac{\pi}{x^2}$$

3 / 5 The leftmost endpoint of R is at $x = 1$ and the rightmost endpoint is at $x = 3$. So the interval of integration is $[1, 3]$.

Now we can express the definite integral in its entirety!

$$\int_1^3 \left(\frac{\pi}{x^2}\right) dx$$

$$= \pi \int_1^3 x^{-2}\, dx$$

4 / 5 Let's evaluate the integral.

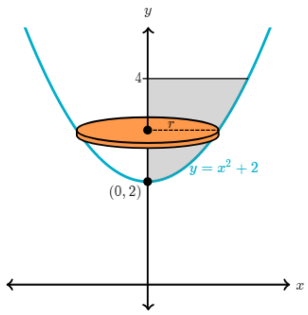
$$\pi \int_1^3 x^{-2}\, dx = \frac{2\pi}{3}$$

[Show calculation.]

5 / 5 **In conclusion, the volume of the solid is $\frac{2\pi}{3}$.**

4)

2 / 5 Let's consider one such slice.



The radius is equal to the distance between the curve $y = x^2 + 2$ and the y-axis. To find it, we need to solve the equation for x:

$$x = \sqrt{y - 2}$$

[Show calculation.]

So, for any y-value, $r(y) = \sqrt{y - 2}$.

Now we can find an expression for the area of the cylinder's base:

$$\pi[r(y)]^2$$

$$= \pi\left(\sqrt{y - 2}\right)^2$$

$$= \pi(y - 2)$$

The bottom endpoint of R is at $y = 2$ and the top endpoint is at $y = 4$. So the interval of integration is $[2, 4]$.

Now we can express the definite integral in its entirety!

$$\int_2^4 [\pi(y - 2)]\, dy$$

$$= \pi \int_2^4 (y - 2)\, dy$$

Let's evaluate the integral.

$$\pi \int_2^4 (y - 2)\, dy = 2\pi$$

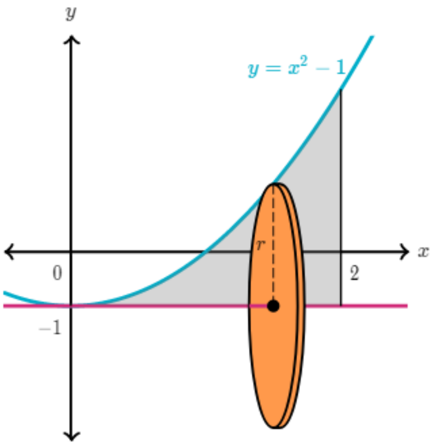
[Show calculation.]

In conclusion, the volume of the solid is 2π.

Lesson 2

1)

Let's consider one such slice.



The radius is equal to the distance between the curve $y = x^2 - 1$ and the line $y = -1$. In other words, for any x-value, this is the equation for $r(x)$:

$$r(x) = (x^2 - 1) - (-1)$$

$$= x^2$$

Now we can find an expression for the area of the cylinder's base:

$$\pi[r(x)]^2$$

$$= \pi\left(x^2\right)^2$$

$$= \pi x^4$$

The leftmost endpoint of R is at $x = 0$ and the rightmost endpoint is at $x = 2$. So the interval of integration is $[0, 2]$.

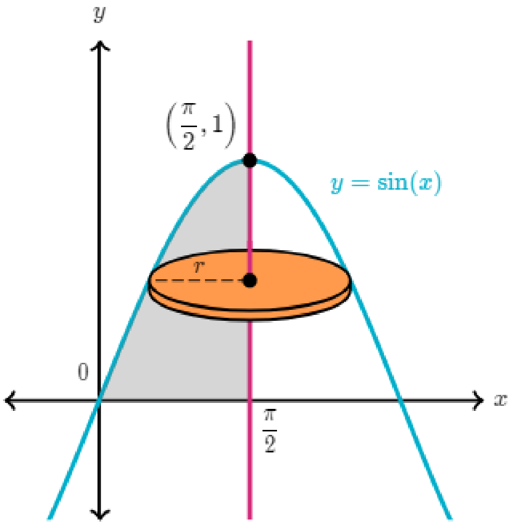
Now we can express the definite integral in its entirety!

$$\int_0^2 \left(\pi x^4\right)\, dx$$

$$= \pi \int_0^2 x^4\, dx$$

$= 32\pi/5$

2)



The radius is equal to the distance between the curve $y = \sin(x)$ and the line $x = \frac{\pi}{2}$. To find it, we need to solve $y = \sin(x)$ for x:

$x = \arcsin(y)$

$\arcsin[\sin(x)]$ will give us the value in the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is the same as $\sin(x)$. Since x is within that range, $\arcsin[\sin(x)]$ is equal to x:

$$y = \sin(x)$$

$$\arcsin(y) = \arcsin[\sin(x)]$$

$$\arcsin(y) = x$$

So, for any y-value, this is the equation for $r(y)$:

$$r(y) = \frac{\pi}{2} - \arcsin(y)$$

$$= \frac{\pi}{2} - \arcsin(y)$$

Now we can find an expression for the area of the cylinder's base:

$$\pi[r(y)]^2$$

$$= \pi\left[\frac{\pi}{2} - \arcsin(y)\right]^2$$

The bottom endpoint of R is at $y = 0$ and the top endpoint is at $y = 1$. So the interval of integration is $[0, 1]$.

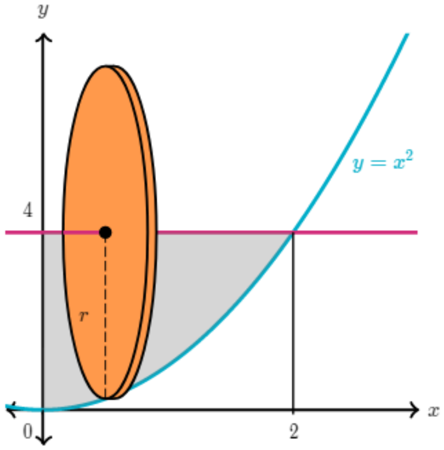
Now we can express the definite integral in its entirety!

$$\int_0^1 \pi\left[\frac{\pi}{2} - \arcsin(y)\right]^2 dy$$

$$= \pi \int_0^1 \left[\frac{\pi}{2} - \arcsin(y)\right]^2 dy$$

3)

Let's consider one such slice.



The radius is equal to the distance between the curve $y = x^2$ and the line $y = 4$. In other words, for any x-value, this is the equation for $r(x)$:

$r(x) = 4 - x^2$

Now we can find an expression for the area of the cylinder's base:

$$\pi[r(x)]^2$$

$$= \pi\left(4 - x^2\right)^2$$

$$= \pi(16 - 8x^2 + x^4)$$

The leftmost endpoint of R is at $x = 0$ and the rightmost endpoint is at $x = 2$. So the interval of integration is $[0, 2]$.

Now we can express the definite integral in its entirety!

$$\int_0^2 \left[\pi(16 - 8x^2 + x^4)\right] dx$$

$$= \pi \int_0^2 (16 - 8x^2 + x^4)\, dx$$

Let's evaluate the integral.

$$\pi \int_0^2 (16 - 8x^2 + x^4)\, dx = \frac{256\pi}{15}$$

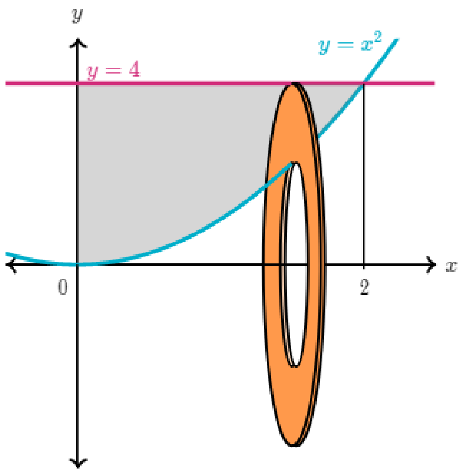
[Show calculation.]

In conclusion, the volume of the solid is $\dfrac{256\pi}{15}$.

Lesson 3

1)

Let's imagine the solid is made out of many thin slices. Each slice is a cylinder with a hole in the middle, much like a washer.



Let the width of each slice be dx, let the radius of the washer, as a function of x, be $r_1(x)$, and let the radius of the hole, as a function of x, be $r_2(x)$.

Then, the volume of each slice is $\pi[(r_1(x))^2 - (r_2(x))^2]\,dx$, and we can sum the volumes of infinitely many such slices with an infinitely small width using a definite integral:

$$\int_a^b \pi[(r_1(x))^2 - (r_2(x))^2]\,dx$$

We call this the *washer method*.

What we now need is to figure out the expressions for $r_1(x)$ and $r_2(x)$ and the interval of integration.

$r_1(x)$ is equal to the distance from line $y = 4$ to the x-axis. So, $r_1(x) = 4$.

$r_2(x)$ is equal to the distance between the curve $y = x^2$ and the x-axis. So, $r_2(x) = x^2$.

Now we can find an expression for the area of the washer's base:

$$\pi[(r_1(x))^2 - (r_2(x))^2]$$

$$= \pi[(4)^2 - (x^2)^2]$$

$$= \pi(16 - x^4)$$

The leftmost endpoint of R is at $x = 0$ and the rightmost endpoint is at $x = 2$. So the interval of integration is $[0, 2]$.

Now we can express the definite integral in its entirety!

$$\int_0^2 \left[\pi\left(16 - x^4\right)\right] dx$$

Let's evaluate the integral.

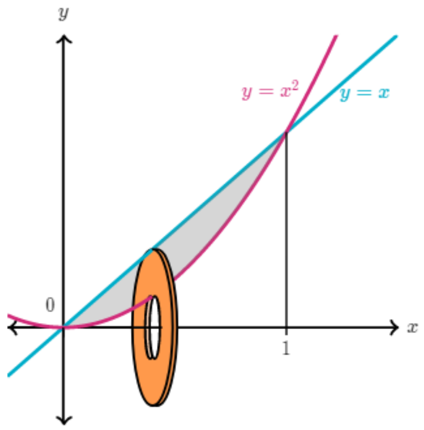
$$\int_0^2 \left[\pi\left(16 - x^4\right)\right] dx = \frac{128\pi}{5}$$

[Show calculation.]

In conclusion, the volume of the solid is $\frac{128\pi}{5}$.

2)

Let's imagine the solid is made out of many thin slices. Each slice is a cylinder with a hole in the middle, much like a washer.



Let the width of each slice be dx, let the radius of the washer, as a function of x, be $r_1(x)$, and let the radius of the hole, as a function of x, be $r_2(x)$.

Then, the volume of each slice is $\pi[(r_1(x))^2 - (r_2(x))^2]\,dx$, and we can sum the volumes of infinitely many such slices with an infinitely small width using a definite integral:

$$\int_a^b \pi[(r_1(x))^2 - (r_2(x))^2]\,dx$$

We call this the *washer method*.

What we now need is to figure out the expressions for $r_1(x)$ and $r_2(x)$ and the interval of integration.

$r_1(x)$ is equal to the distance from line $y = x$ to the x-axis. So, $r_1(x) = x$.

$r_2(x)$ is equal to the distance between the curve $y = x^2$ and the x-axis. So, $r_2(x) = x^2$.

Now we can find an expression for the area of the washer's base:

$$\pi[(r_1(x))^2 - (r_2(x))^2]$$

$$= \pi[(x)^2 - (x^2)^2]$$

$$= \pi(x^2 - x^4)$$

The leftmost endpoint of R is at $x = 0$ and the rightmost endpoint is at $x = 1$. So the interval of integration is $[0, 1]$.

Now we can express the definite integral in its entirety!

$$\int_0^1 \left[\pi\left(x^2 - x^4\right)\right] dx$$

Let's evaluate the integral.

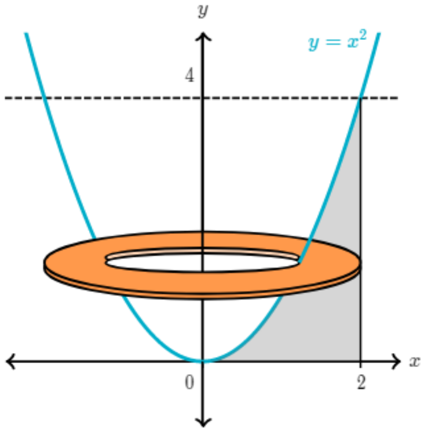
$$\int_0^1 \left[\pi\left(x^2 - x^4\right)\right] dx = \frac{2\pi}{15}$$

[Show calculation.]

In conclusion, the volume of the solid is $\frac{2\pi}{15}$.

3)

Let's imagine the solid is made out of many thin slices. Each slice is a cylinder with a hole in the middle, much like a washer.



Let the width of each slice be dy, let the radius of the washer, as a function of y, be $r_1(y)$, and let the radius of the hole, as a function of y, be $r_2(y)$.

Then, the volume of each slice is $\pi[(r_1(y))^2 - (r_2(y))^2]\,dy$, and we can sum the volumes of infinitely many such slices with an infinitely small width using a definite integral:

$$\int_a^b \pi[(r_1(y))^2 - (r_2(y))^2]\,dy$$

We call this the *washer method*.

$r_1(y)$ is equal to the distance from line $x = 2$ to the y-axis. So, $r_1(y) = 2$.

$r_2(y)$ is equal to the distance from the curve $y = x^2$ to the y-axis. To find it, we need to solve the equation for x:

$x = \sqrt{y}$

[Show calculation.]

So, $r_2(y) = \sqrt{y}$.

Now we can find an expression for the area of the washer's base:

$$\pi[(r_1(y))^2 - (r_2(y))^2]$$

$$= \pi[(2)^2 - (\sqrt{y})^2]$$

$$= \pi(4 - y)$$

The bottom endpoint of R is at $y = 0$ and the top endpoint is at $y = 4$. So the interval of integration is $[0, 4]$.

Now we can express the definite integral in its entirety!

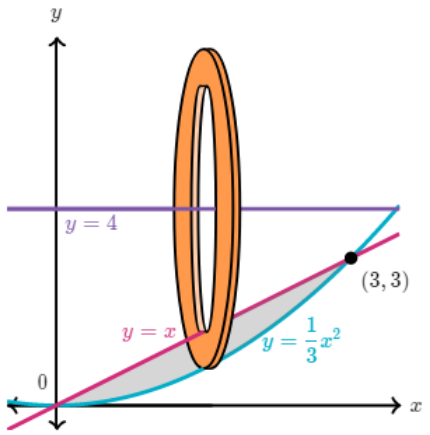
$$\int_0^4 [\pi\,(4 - y)]\,dy$$

Let's evaluate the integral.

$$\int_0^4 [\pi\,(4 - y)]\,dy = 8\pi$$

4)

Let's imagine the solid is made out of many thin slices. Each slice is a cylinder with a hole in the middle, much like a washer.



Let the thickness of each slice be dx, let the radius of the washer, as a function of x, be $r_1(x)$, and let the radius of the hole, as a function of x, be $r_2(x)$.

Then, the volume of each slice is $\pi[(r_1(x))^2 - (r_2(x))^2]\,dx$, and we can sum the volumes of infinitely many such slices with an infinitely small thickness using a definite integral:

$$\int_a^b \pi[(r_1(x))^2 - (r_2(x))^2]\,dx$$

This is called the *washer method*.

What we now need is to figure out the expressions of $r_1(x)$ and $r_2(x)$, and the interval of integration.

$r_1(x)$ is equal to the distance between the curve $y = \frac{1}{3}x^2$ and the line $y = 4$. So, $r_1(x) = 4 - \frac{1}{3}x^2$.

$r_2(x)$ is equal to the distance between the line $y = x$ and the line $y = 4$. So, $r_2(x) = 4 - x$.

Now we can find an expression for the area of the washer's base:

$$\pi[(r_1(x))^2 - (r_2(x))^2]$$

$$= \pi\left[\left(4 - \frac{1}{3}x^2\right)^2 - (4 - x)^2\right]$$

$$= \pi\left[\left(16 - \frac{8}{3}x^2 + \frac{1}{9}x^4\right) - (16 - 8x + x^2)\right]$$

$$= \pi\left(\frac{1}{9}x^4 - \frac{11}{3}x^2 + 8x\right)$$

The leftmost endpoint of R is at $x = 0$ and the rightmost endpoint is at $x = 3$. So the interval of integration is $[0, 3]$.

Now we can express the definite integral in its entirety!

$$\int_0^3 \left[\pi\left(\frac{1}{9}x^4 - \frac{11}{3}x^2 + 8x\right)\right] dx$$

$$= \pi \int_0^3 \left(\frac{1}{9}x^4 - \frac{11}{3}x^2 + 8x \right) \, dx$$

Let's evaluate the integral.

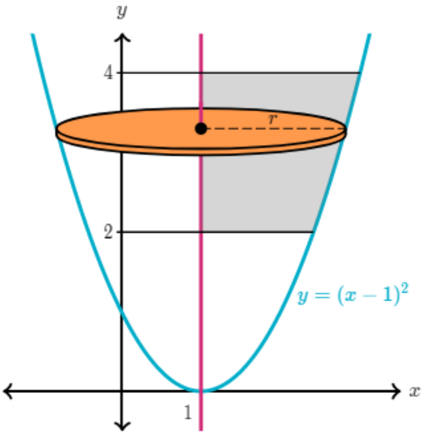
$$\pi \int_0^3 \left(\frac{1}{9}x^4 - \frac{11}{3}x^2 + 8x \right) \, dx = \frac{42\pi}{5}$$

[Show calculation.]

In conclusion, the volume of the solid is $\frac{42\pi}{5}$.

Quiz Solutions

1)



The radius is equal to the distance between the curve $y = (x-1)^2$ and the line $x = 1$. To find it, we need to solve $y = (x-1)^2$ for x:

$x = \sqrt{y} + 1$

[Show calculation.]

So, for any y-value, this is the equation for $r(y)$:

$$r(y) = (\sqrt{y} + 1) - (1)$$

$$= \sqrt{y}$$

Now we can find an expression for the area of the cylinder's base:

$$\pi[r(y)]^2$$

$$= \pi\left(\sqrt{y}\right)^2$$

$$= \pi y$$

The bottom endpoint of R is at $y = 2$ and the top endpoint is at $y = 4$. So the interval of integration is $[2, 4]$.

Now we can express the definite integral in its entirety!

$$\int_2^4 (\pi y)\, dy$$

$$= \pi \int_2^4 y\, dy$$

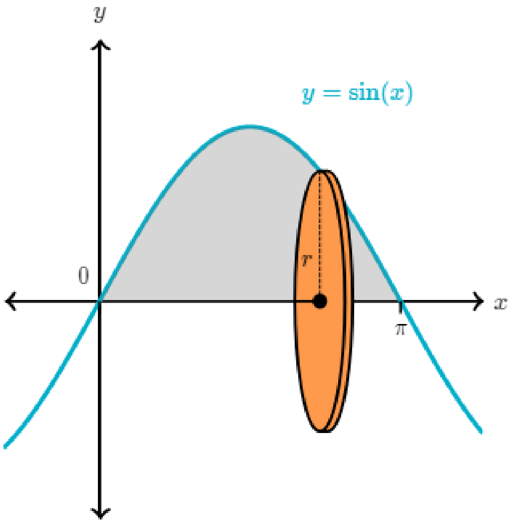
Let's evaluate the integral.

$$\pi \int_2^4 y\, dy = 6\pi$$

[Show calculation.]

In conclusion, the volume of the solid is 6π.

2)



The radius is equal to the distance between the curve $y = \sin(x)$ and the x-axis. In other words, for any x-value, $r(x) = \sin(x)$.

Now we can find an expression for the area of the cylinder's base:

$$\pi[r(x)]^2$$

$$= \pi\,(\sin(x))^2$$

$$= \pi \cdot \sin^2(x)$$

The leftmost endpoint of R is at $x = 0$ and the rightmost endpoint is at $x = \pi$. So the interval of integration is $[0, \pi]$.

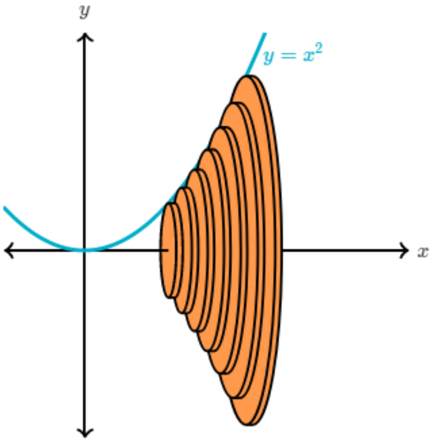
Now we can express the definite integral in its entirety!

$$\int_0^\pi \left[\pi \cdot \sin^2(x)\right] dx$$

$$= \pi \int_0^\pi \sin^2(x)\, dx$$

3)

Let's imagine the solid is made out of many thin slices.



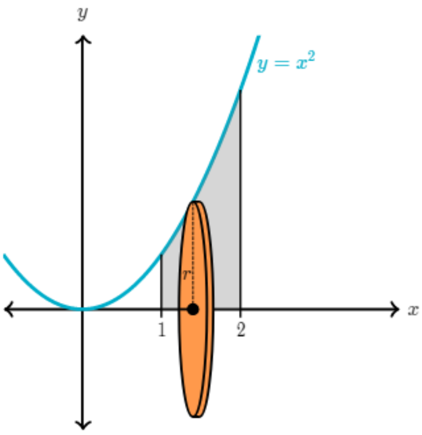
Each slice is a *cylinder*. Let the thickness of each slice be dx and let the radius of the base, as a function of x, be $r(x)$.

Then, the volume of each slice is $\pi[r(x)]^2\,dx$, and we can sum the volumes of infinitely many such slices with an infinitely small thickness using a definite integral:

$$\int_a^b \pi[r(x)]^2\,dx$$

This is called the *disc method*.

Let's consider one such slice.



The radius is equal to the distance between the curve $y = x^2$ and the x-axis. In other words, for any x-value, $r(x) = x^2$.

Now we can find an expression for the area of the cylinder's base:

$$\pi[r(x)]^2$$

$$= \pi\left(x^2\right)^2$$

$$= \pi x^4$$

The leftmost endpoint of R is at $x = 1$ and the rightmost endpoint is at $x = 2$. So the interval of integration is $[1, 2]$.

Now we can express the definite integral in its entirety!

$$\int_1^2 (\pi x^4)\, dx$$

$$= \pi \int_1^2 x^4 dx$$

Let's evaluate the integral.

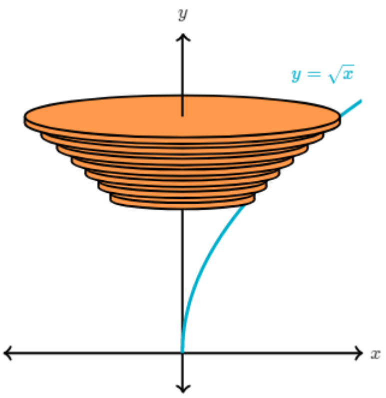
$$\pi \int_1^2 x^4\, dx = \frac{31\pi}{5}$$

[Show calculation.]

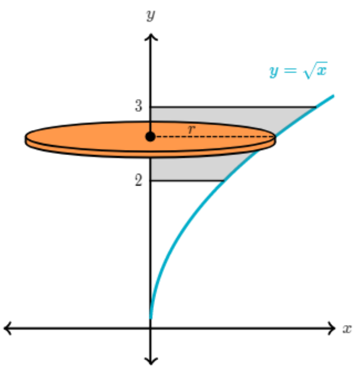
In conclusion, the volume of the solid is $\frac{31\pi}{5}$.

4)

Let's imagine the solid is made out of many thin slices.



Notice the slices are *horizontal*, because we are rotating R about the y-axis. Each slice is a *cylinder*. Let the thickness of each slice be dy and let the radius of the base, as a function of y, be $r(y)$.



The radius is equal to the distance between the curve $y = \sqrt{x}$ and the y-axis. To find it, we need to solve the equation for x:

$x = y^2$

[Show calculation.]

So, for any y-value, $r(y) = y^2$.

Now we can find an expression for the area of the cylinder's base:

$$\pi[r(y)]^2$$

$= \pi \left(y^2\right)^2$

$= \pi y^4$

The bottom endpoint of R is at $y = 2$ and the top endpoint is at $y = 3$. So the interval of integration is $[2, 3]$.

Now we can express the definite integral in its entirety!

$$\int_2^3 \left(\pi y^4\right) dy$$

$$= \pi \int_2^3 y^4 \, dy$$

Let's evaluate the integral.

$$\pi \int_2^3 \left(y^4\right) \, dy = \frac{211\pi}{5}$$

[Show calculation.]

In conclusion, the volume of the solid is $\frac{211\pi}{5}$.