GreatHearts®

Calculus I

April 27 – May 1

Time Allotment: 40 minutes per day

Student Name: ________________________________

Teacher Name: ________________________________

Calculus
April 27 – May 1

GreatHearts®

Packet Overview

Date	Objective(s)	Page Number
Monday, April 27	Review: Finding Area Between Two Functions	2-4
Tuesday, April 28	Review: Integrating Using Disc Method	5-7
Wednesday, April 29	Review: Integrating Using Washer Method	8-9
Thursday, April 30	Calculating Work with Variable Force	10-11
Friday, May 1	Quiz	12-14

Additional Notes: Dear Students: this week we're going to review using integrals to enter into the third dimension and find the volumes of solids. A solid week of practice will help you understand more deeply the consequences of the kinds of integration you're doing in three dimensions.

Though not required to complete these assignments, Khan Academy's AP Calculus AB series of videos are a helpful resource for supplemental learning. We'll be going over the chapter on Applications of Integrals. If you're having trouble seeing the 3-D pictures on my handwritten notes, they follow closely the videos titled "Volumes with cross sections" in that Applications of Integrals section. This week, though, we will largely be relying on our red textbook for examples and exercises. Get ready for another fun week!

Academic Honesty

I certify that I completed this assignment independently in accordance with the GHNO Academy Honor Code.

Student signature:

____

I certify that my student completed this assignment independently in accordance with the GHNO Academy Honor Code.

Parent signature:

____

Calculus
April 27 – May 1

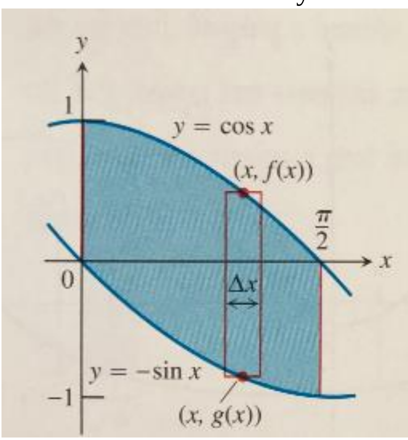
Monday, April 27

Calculus Unit: Applications of Integrals
Lesson 1: Finding area between curves.

Objective: Be able to do this by the end of this lesson.
1. Given two curves, integrate to find the area between them.
2. Determine whether to integrate along the x-axis or the y-axis.

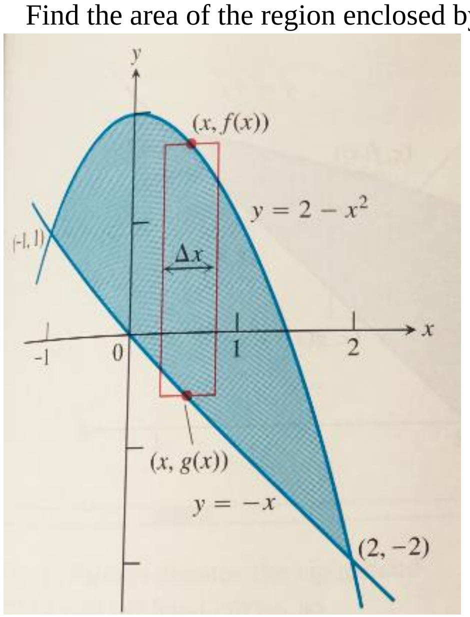
Introduction to Lesson 1
Let's do some review of finding area between curves. I want to make sure you've mastered this skill before we move on to other topics in calculus.

1. Find the area between y = cos x and y = -sin x from 0 to π/2.

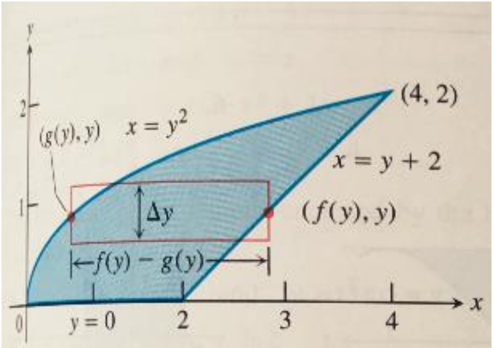


Try it out on your own. You can check your answer p. 440 in your textbook.

2. Now let's try one where you have to integrate between intersection points.
Find the area of the region enclosed by the parabola $y = 2 – x^2$ and the line y = -x. (solution on p. 441)

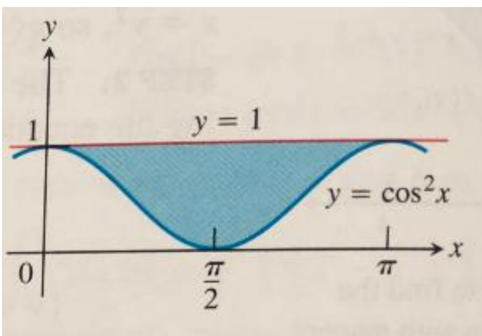


3. For this next exercise, you could integrate along the x-axis, but it would be easier to integrate along the y-axis, and make the slices have thickness dy. Find the area bounded by $y = \sqrt{x}$ and y = x – 2.
Check your answer on p. 443.

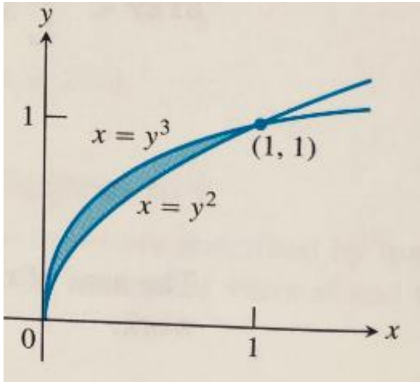


Now let's do some more practice! Find the area of the shaded regions.

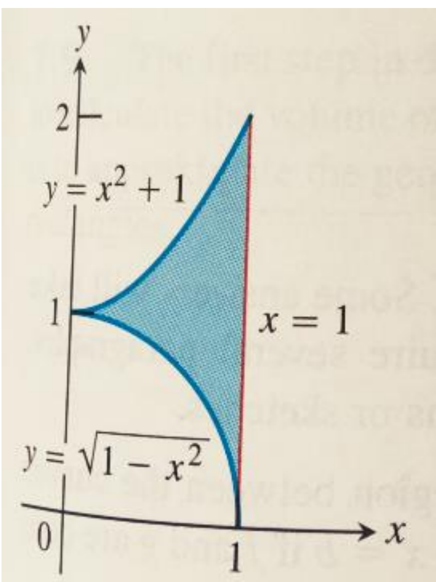
4.



5.



6.



Calculus
April 27 – May 1

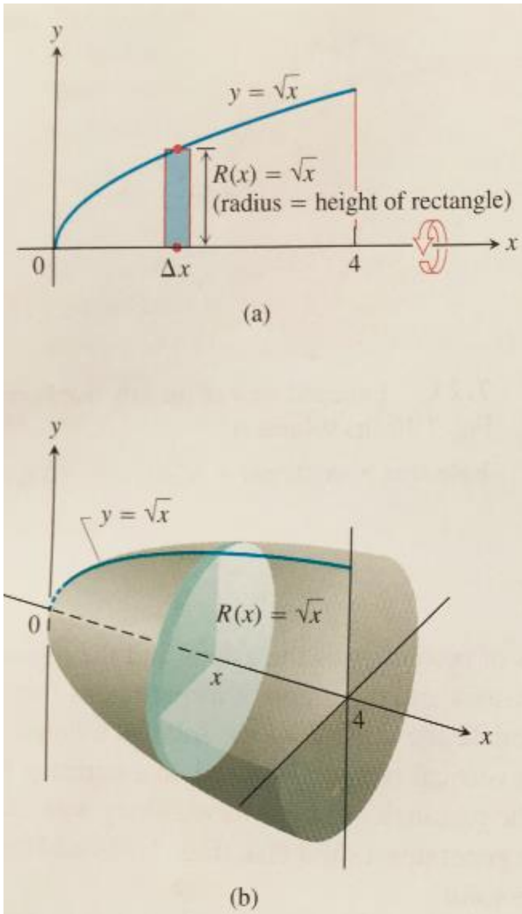
Tuesday, April 28

Calculus Unit: Applications of Integrals
Lesson 2: Rotating using the disc method.

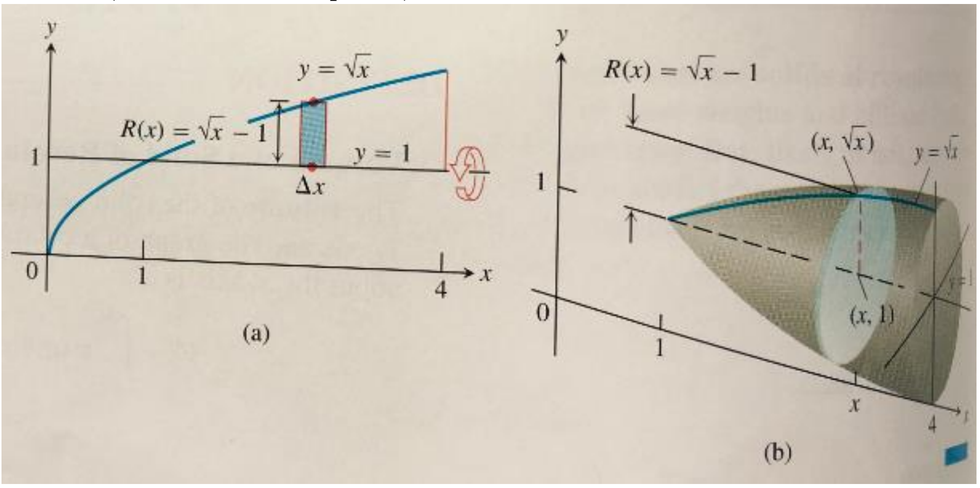
Objective: Be able to do this by the end of this lesson.
1. Find volume using disc method while accounting for the axis of rotation (be it the x-axis, y-axis, or a different axis of rotation..

Introduction to Lesson 2
Today we'll do some extra practice of using the disc method for rotating objects about different axes.

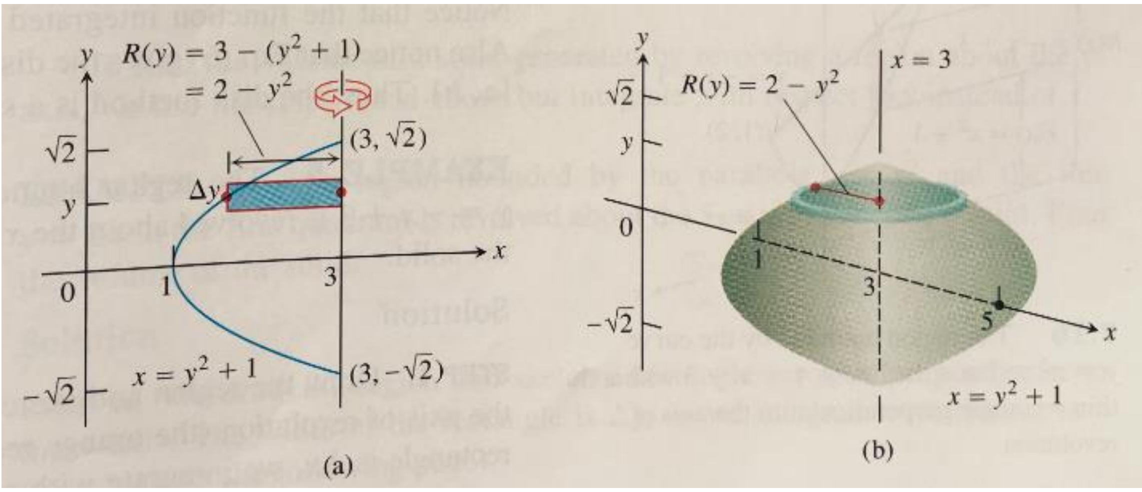
1. Let's take the curve $y = \sqrt{x}$ between $0 \le x \le 4$, and revolve the function about the x-axis to generate the solid below. Find the volume of that solid (and then check your answer on p. 448).



2. Now let's find the volume when the same function is rotated about the y = 1 axis, with the same interval. (check solution on p. 448)



3. And now one where you have to use dy thicknesses and rotate about the vertical line x = 3 (check answer on p. 449).



On a separate sheet of paper, do numbers 1, 3, and 9 on p. 453 (Section 7-2).

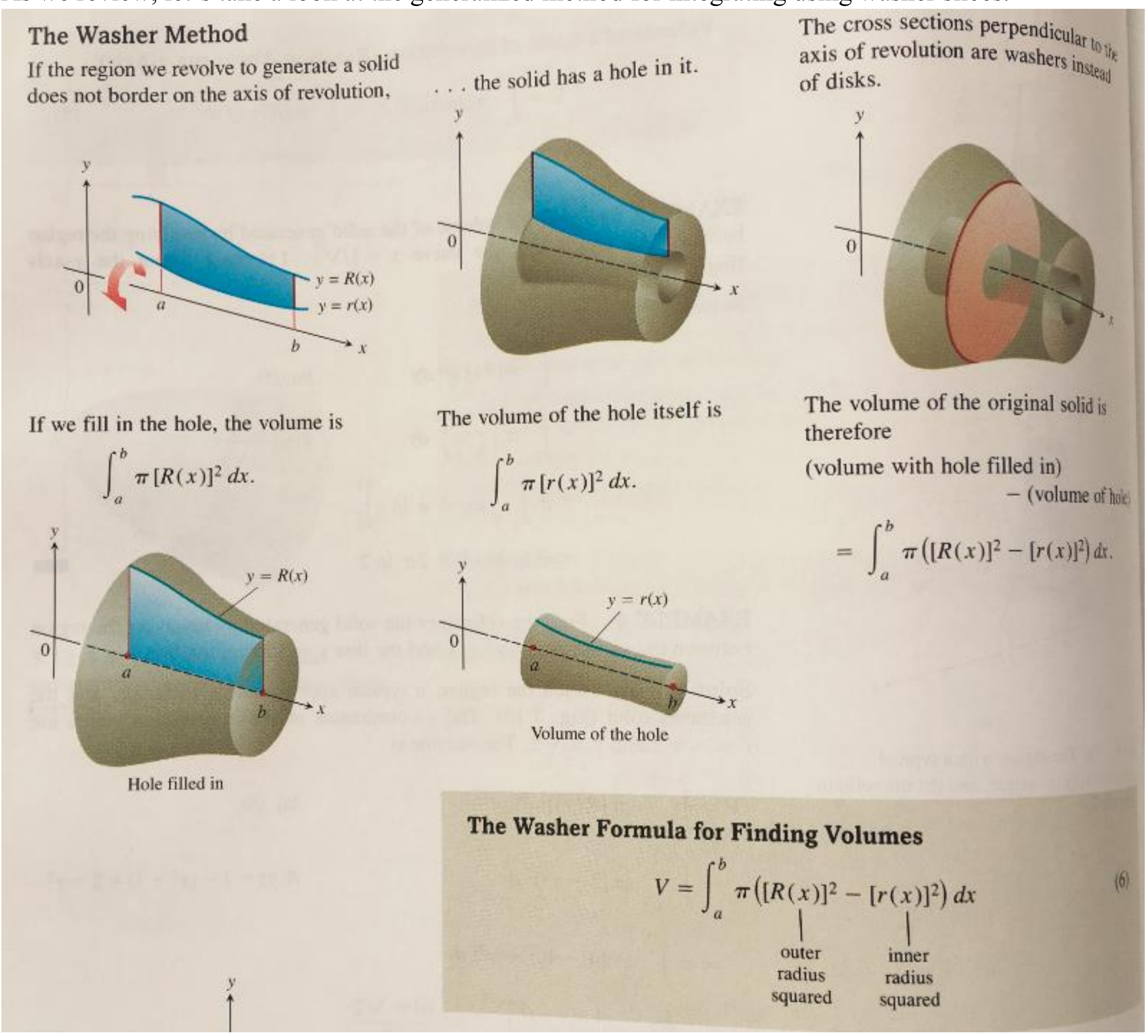
<u>Wednesday April 29</u>

Calculus Unit: Applications of Integrals
Lesson 3: Finding Volume of Using the Washer Method

Objective: Be able to do this by the end of this lesson.
1. Review using the washer method when integrating a rotation about the x-axis or y-axis.
2. Review using the washer method when integrating a rotation about an arbitrary vertical or horizontal line.

Introduction to Lesson 3
Today we're going to review the washer method. As you remember, the key is finding the two radii of the washer and then subtracting the bigger circle area from the smaller circle area.

As we review, let's take a look at the generalized method for integrating using washer slices.



The Washer Method

If the region we revolve to generate a solid does not border on the axis of revolution, . . . the solid has a hole in it. The cross sections perpendicular to the axis of revolution are washers instead of disks.

If we fill in the hole, the volume is

$$\int_a^b \pi [R(x)]^2 \, dx.$$

The volume of the hole itself is

$$\int_a^b \pi [r(x)]^2 \, dx.$$

The volume of the original solid is therefore

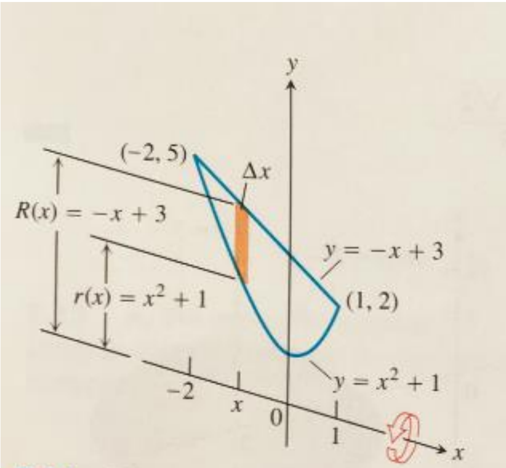
(volume with hole filled in) − (volume of hole)

$$= \int_a^b \pi \left([R(x)]^2 - [r(x)]^2\right) dx.$$

The Washer Formula for Finding Volumes

$$V = \int_a^b \pi \left(\underbrace{[R(x)]^2}_{\text{outer radius squared}} - \underbrace{[r(x)]^2}_{\text{inner radius squared}}\right) dx \qquad (6)$$

1. Now let's work a washer problem: The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the out and inner radii, then the surface area of the washer, then volume of the washer, and finally the volume of the solid. (check your answer on p. 451).



7.16 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ with a thin rectangle perpendicular to the axis of revolution.

2. This is a washer problem that requires you to integrate with respect to y and use dy slices. Try to draw the graph on your own. If you need help, you can follow the steps on p. 452.

The region bounded by the parabola y = x2 and the line y = 2x in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

On a separate sheet of paper, do exercises 25, 27, and 37.

Calculus
April 27 – May 1

Thursday, April 30

Calculus Unit: Applications of Integrals
Lesson 4: Applying Integrals to Work

Objective: Be able to do this by the end of this lesson.
1. Demonstrate mastery of disc and washer methods of integration.

Introduction to Lesson 4
Today we're going to apply integrals to work. In physics class, you remember that work is equal to the applied force on an object times the distance the object moves times the cosine of the angle between them. What if the force you are trying to apply to an object is changing? For instance, say the higher you pull a leaky bucket of water up on a pulley, the less force you have to exert on a rope because water is leaking out of it. We only worked problems in physics class where there is a constant force. We can use integrals to solve work problems when the force varies in this way.

1. Say the work done by a force of $F(x) = 1/x^2$ along the x-axis from x = 1m to x = 10m. Set up the integral, solve the problem, and check your steps on p. 473.

2. Now there's a leaky 5-lb bucket of water lifted from the ground into the air. It's pulled by a 20ft. rope at a constant speed (see Figure 7.45 on p. 473). The rope weighs 0.08 lb/ft. The bucket starts with 2 gal of water (16 lb) and leads at a constant rate, emptying when it reaches the top.

a. Calculate the force required to lift just the water. Remember, the force required to lift the water is just equal to the water's weight, which changes at a constant rate from 16 lb to 0 lb over 20 ft. Check your answer on p. 473.

b. Calculate the work done when lifting the water and bucket together. Calculating the bucket is easy, it's just the bucket's weight times the distance it moves.

c. Calculate the work done when lifting the water, bucket, and the rope. For the rope, you'll need to set up an integral again. Think about how you're lifting less rope as you pull it through the pulley. Check the equation on p. 474 if you need help setting this up.

2. Finally, do Exercise 1 on p. 477.

GreatHearts®

Friday, May 1

Calculus Unit: Applications of Integrals
Lesson 1: Quiz

Objective: Be able to do this by the end of this lesson.
1. Demonstrate mastery of finding area between two functions and rotating using disc and washer methods of integration.

Calculus – Quiz on Area Between Curves, Disc Method, and Washer Method

Name: ___________________________

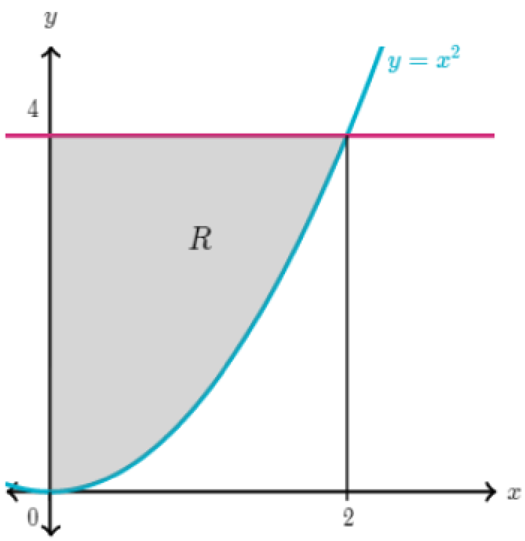
1) Find the area between $y = x^2 - 2$ and $y = 2$.

2) Find the area between $y = x^4$ and $y = 8x$.

3) Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{9 - x^2}$ and $y = 0$ about the x-axis (use dx slices).

4)

Let R be the region enclosed by the y-axis, the line $y = 4$ and the curve $y = x^2$.



A solid is generated by rotating R about the line $y = 4$.

What is the volume of the solid?
Give an exact answer in terms of π.

Calculus
April 27 – May 1

Answer Key:

Lesson 1:

4)

For the sketch given, $a = 0$, $b = \pi$; $f(x) - g(x) = 1 - \cos^2 x = \sin^2 x$;

$$A = \int_0^{\pi} \sin^2 x\,dx = \int_0^{\pi} \frac{(1-\cos 2x)}{2}dx = \frac{1}{2}\int_0^{\pi}(1-\cos 2x)dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}[\pi - 0 - 0 + 0] = \frac{\pi}{2}$$

5)

For the sketch given, $c = 0$, $d = 1$; $f(y) - g(y) = y^2 - y^3$;

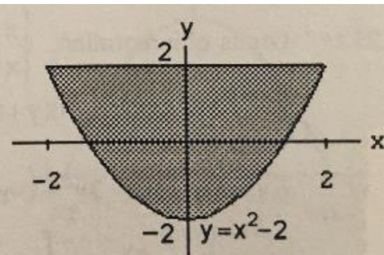
$$A = \int_0^1 (y^2 - y^3)dy = \int_0^1 y^2 dy - \int_0^1 y^3 dy = \left[\frac{y^3}{3}\right]_0^1 - \left[\frac{y^4}{4}\right]_0^1 = \frac{(1-0)}{3} - \frac{(1-0)}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

6)

$a = -2$, $b = 2$

$f(x) - g(x) = 2 - (x^2 - 2) = 4 - x^2$

$$A = \int_{-2}^{2} (4 - x^2)dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 8 - \frac{8}{3} + 8 - \frac{8}{3} = 2\cdot\left(\frac{24}{3} - \frac{8}{3}\right) = \frac{32}{3}$$



Graph 7.1.13

Calculus
April 27 – May 1

GreatHearts

Lesson 2

1) $2\pi/3$

3) $4 - \pi$

9) $32\pi/5$

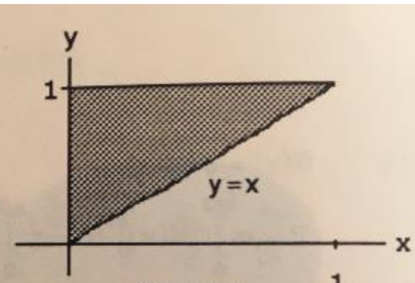
Lesson 3

25)

$r(x) = x; \quad R(x) = 1$

$$V = \int_0^1 \pi\left[[R(x)]^2 - [r(x)]^2\right]dx = \int_0^1 \pi\left[1 - x^2\right]dx = \pi\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \pi\left[\left(1 - \frac{1}{3}\right) - 0\right] = \frac{2\pi}{3}$$



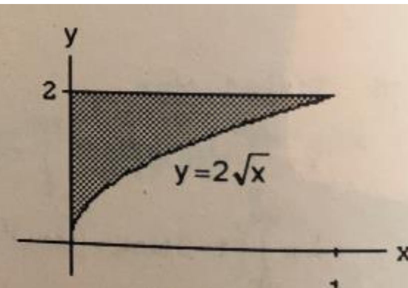
Graph 7.2.25

27)

$r(x) = 2\sqrt{x}, \quad R(x) = 2$

$$V = \int_0^1 \pi\left[[R(x)]^2 - [r(x)]^2\right]dx = \pi\int_0^1 [4 - 4x]dx = 4\pi\left[x - \frac{x^2}{2}\right]_0^1$$

$$= 4\pi\left(1 - \frac{1}{2}\right) = 2\pi$$



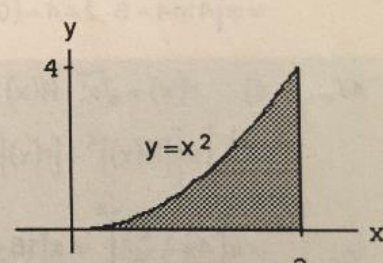
Graph 7.2.27

37)

$R(y) = 2; \quad r(y) = \sqrt{y}$

$$V = \int_0^4 \pi\left[[R(y)]^2 - [r(y)]^2\right]dy = \pi\int_0^4 [4 - y]dy = \pi\left[4y - \frac{y^2}{2}\right]_0^4$$

$$= \pi[16 - 8] = 8\pi$$



Graph 7.2.37

Calculus
April 27 – May 1

Lesson 4

1)

The force required to lift the water is equal to the water's weight, which varies steadily from 40 lb to 0 lb over the 20-ft lift. When the bucket is x ft off the ground, the water weighs:

$F(x) = 40\cdot\left(\frac{20-x}{20}\right) = 40\left(1-\frac{x}{20}\right) = 40-2x$ lb. The work done is:

$$W = \int_a^b F(x)dx = \int_0^{20}(40-2x)dx = \left[40x-x^2\right]_0^{20} = 40(20)-20^2 = 800-400 = 400 \text{ ft}\cdot\text{lb}.$$

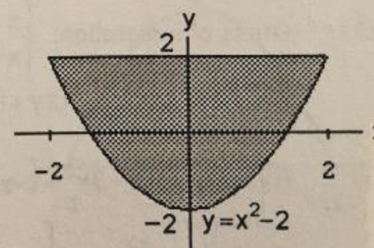
Quiz Solutions

1)

$a=-2,\ b=2$

$f(x)-g(x) = 2-\left(x^2-2\right) = 4-x^2$

$$A = \int_{-2}^{2}\left(4-x^2\right)dx = \left[4x-\frac{x^3}{3}\right]_{-2}^{2} = 8-\frac{8}{3}+8-\frac{8}{3} = 2\cdot\left(\frac{24}{3}-\frac{8}{3}\right) = \frac{32}{3}$$



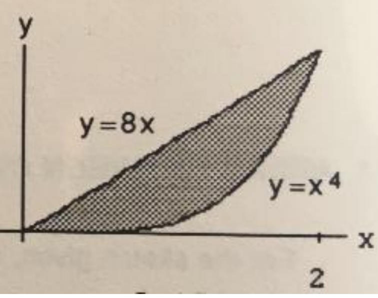
Graph 7.1.13

2)

$a=0,\ b=2$

$f(x)-g(x) = 8x-x^4$

$$A = \int_0^2\left(8x-x^4\right)dx = \left[\frac{8x^2}{2}-\frac{x^5}{5}\right]_0^2 = 16-\frac{32}{5} = \frac{80-32}{5} = \frac{48}{5}$$

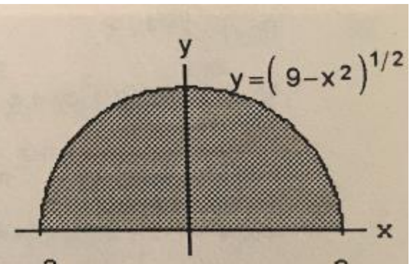


3)

$R(x) = \sqrt{9-x^2}$

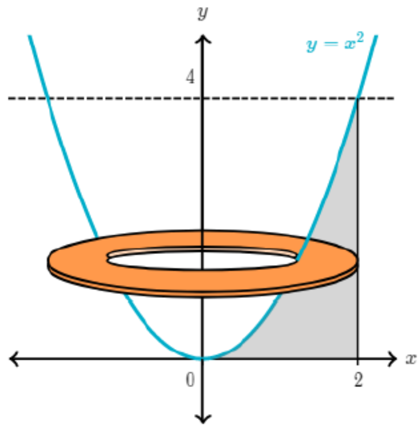
$$V = \int_{-3}^{3}\pi\left[R(x)\right]^2dx = \pi\int_{-3}^{3}\left(9-x^2\right)dx = \pi\left[9x-\frac{x^3}{3}\right]_{-3}^{3} = 2\pi\left[9(3)-\frac{27}{3}\right]$$

$$= 2\cdot\pi\cdot18 = 36\pi$$



Graph 7.2.11

4)



Let the width of each slice be dy, let the radius of the washer, as a function of y, be $r_1(y)$, and let the radius of the hole, as a function of y, be $r_2(y)$.

Then, the volume of each slice is $\pi[(r_1(y))^2 - (r_2(y))^2]\,dy$, and we can sum the volumes of infinitely many such slices with an infinitely small width using a definite integral:

$$\int_a^b \pi[(r_1(y))^2 - (r_2(y))^2]\,dy$$

We call this the *washer method*.

$r_1(y)$ is equal to the distance from line $x = 2$ to the y-axis. So, $r_1(y) = 2$.

$r_2(y)$ is equal to the distance from the curve $y = x^2$ to the y-axis. To find it, we need to solve the equation for x:

$$x = \sqrt{y}$$

So, $r_2(y) = \sqrt{y}$.

Now we can find an expression for the area of the washer's base:

$$\pi[(r_1(y))^2 - (r_2(y))^2]$$

$$= \pi[(2)^2 - (\sqrt{y})^2]$$

$$= \pi(4 - y)$$

The bottom endpoint of R is at $y = 0$ and the top endpoint is at $y = 4$. So the interval of integration is $[0, 4]$.

Now we can express the definite integral in its entirety!

$$\int_0^4 [\pi\,(4 - y)]\,dy$$

Let's evaluate the integral.

$$\int_0^4 [\pi\,(4 - y)]\,dy = 8\pi$$