GreatHearts®

Calculus I

May 4 – May 8

Time Allotment: 40 minutes per day

Student Name: ________________________________

Teacher Name: ________________________________

Packet Overview

Date	Objective(s)	Page Number
Monday, May 4	Review Exponential Functions	2-3
Tuesday, May 5	Taking the Derivative of Exponential Functions	4-5
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Additional Notes: Dear Students: We are at our journey's end, the home stretch! The last topic we'll study in this class is exponential functions. You've been introduced to them in previous math classes, but in calculus, we're going to go deeper into its essence. There are many amazing applications of these functions that we'll look at in detail. These applications will serve to help us see better what an exponential function is, and what is so special about them.

Academic Honesty

I certify that I completed this assignment independently in accordance with the GHNO Academy Honor Code.

Student signature:

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I certify that my student completed this assignment independently in accordance with the GHNO Academy Honor Code.

Parent signature:

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Monday, May 4

Calculus Unit: Transcendental Functions
Lesson 1: Exponential Functions

Objective: Be able to do this by the end of this lesson.
1. Graph basic exponential functions.
2. Identify shape of graph when base is greater than 1 and less than 1.
3. Apply laws of exponents to simplify expressions.

Introduction to Lesson 1
Today we're going to do a review of exponential functions. What is an exponential function? How is it different than the functions we've studied so far? What rules govern exponents, and what do their graphs look like? We'll get solid on all of those things in this lesson.

1. What happens when we take a function and raise a positive constant to a variable power? For instance, take $y = 3^x$. Sketch a graph of what that looks like below.

What's neat is that in graphing a continuous function, you included both rational and irrational numbers (see Figure 6.1 on p. 348). These are called exponential functions.

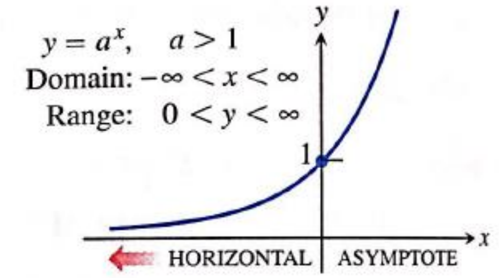
For any constant $a > 0$, the continuous function f defined by the equation

$$f(x) = a^x$$

is the **exponential function to the base** $\boldsymbol{a}$. The positive number a is the **base** of the function.

Figure 6.2 shows the graph of $y = a^x$ for various positive values of a. The only constant exponential function is $y = 1^x$. The other exponential functions either increase or decrease. Their range is $(0, \infty)$ and their graphs all have the x-axis as a horizontal asymptote. As the graphs in the figure suggest, the values of exponential functions *are always positive*. They have the following limits as $x \to \pm\infty$.

2. This happens when a > 1, as shown below.



What if a is between 0 and 1, that is $0 < a < 1$? Sketch that graph below, and include its limit as a goes to both positive and negative infinity, as well as its domain and range.

3. Exponential functions ______________________ (increase/decrease) if $0 < a < 1$, and they ________________________(increase/decrease) if $a > 1$.

4. For a review of the laws of exponents, copy Table 6.2 on p. 348 below.

1.

2.

3.

4.

5.

5. Apply the laws of exponents to solve the following problems:

1. $3^{1.1} \cdot 3^{0.7}$

2. $\dfrac{(\sqrt{10})^3}{\sqrt{10}}$

3. $\left(5^{\sqrt{2}}\right)^{\sqrt{2}}$

4. $7^{\pi} \cdot 8^{\pi}$

5. $\left(\dfrac{4}{9}\right)^{1/2}$

Do Exercises 1, 3, 17, 19, and 21 on p. 353.

Tuesday, May 5

Calculus Unit: Transcendental Functions
Lesson 2: Taking the Derivative of Exponential functions.

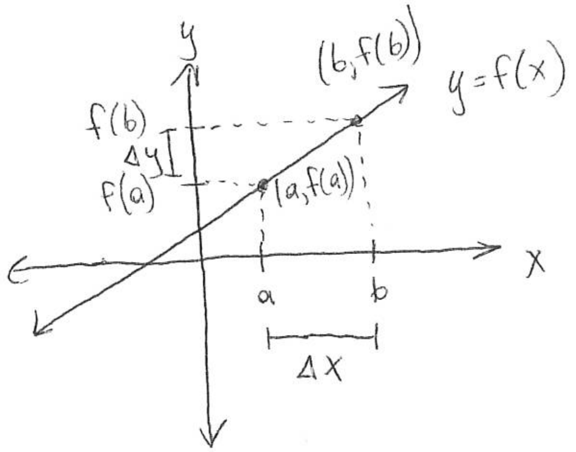
Objective: Be able to do this by the end of this lesson.
1. Write the definition of the derivative using limit theory.
2. Find the derivative of exponential functions.
3. Find a base where the derivative of the function converges on the value of the original function.

Introduction to Lesson 2
Today we're going to learn how to take the derivative of exponential functions.

Let's do a quick review of the definition of the derivative.

See if you can write the definition of the derivative below from memory using limit notation. If you can't remember, it's okay. Flip the page for a review!

Definition of the Derivative - A Quick Review!



How do we find the slope of this line?

Need our slope formula : $y = mx + b$

Plus we need 2 points,
$(a, f(a))$ and $(b, f(b))$

What is the slope formula? slope $= \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}$

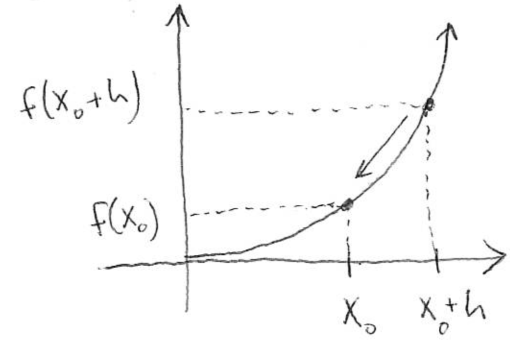
For our particular function, $\Delta y = f(b) - f(a)$

and

$\Delta x = b - a$

Putting them together, $\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$.

How can we find the slope at a single point? because for a function like $y = x^2$, there are an infinite number of slopes. We could find the average slope over an interval, but recall how we can use the derivative to find the slope at one point.



What will the slope be if we found the slope of the secant line between these 2 points?

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0}$$

Simplified...

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{h}$$

What if we took the limit of the function as h goes to zero? This would move the point $(x_0+h, f(x_0+h))$ infintitesimaly close to $(x_0, f(x_0))$, so close that you can't tell that they're different points.

$$f'(x) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

This is the derivative of x, or the slope at the single point, x.

Now let's see how we can apply the definition of the derivative to exponential functions.

1. Start here: $\frac{d}{dx}(a^x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$

End here: $\frac{d}{dx}(a^x) = L \cdot a^x$

Using the laws of exponents, write out all the steps above, and annotate your steps to explain better the moves you are making.

2. That's a nice result! Let's practice taking some derivatives and estimating their values.

$\frac{d}{dx}(2^x)$ will be equal to some constant, L, times 2^x. To figure out what L is, graph $y = (\frac{2^h - 1}{h})$ Find the y-intercept. You'll run into a problem, though, because when you plug in 0 for h, the function will be undefined at the y-axis (it's a removable discontinuity just at that one point – the rest of the function is continuous). So you'll have to plug in values that are slightly greater than 0, i.e. 0.1, 0.01, 0.001, and also values that are slightly less than 0, i.e. -0.1, -.01, -0.001. Try it out and see what you get. Remember, you can also use the table method on your calculator or just plug the values into the function. Check the bottom of p. 349 for the answer.

3. Now, on a separate sheet of paper, do the same for the derivative of 2.5^x and 3^x. What do you notice about L, the constant term out front? Try a few more values between 2.5 and 3. Is there a number that could be used as the base term in an exponential function where L = 1? Wouldn't that be strange and miraculous?! We've never seen anything like that before in this class. We could take the derivative of that exponential function, and its derivative would be exactly the same as the original function. Wow! Could it be!? Try out a few more numbers, maybe 2.6, 2.7, and 2.8, and see which one gives you an L value closest to 1.

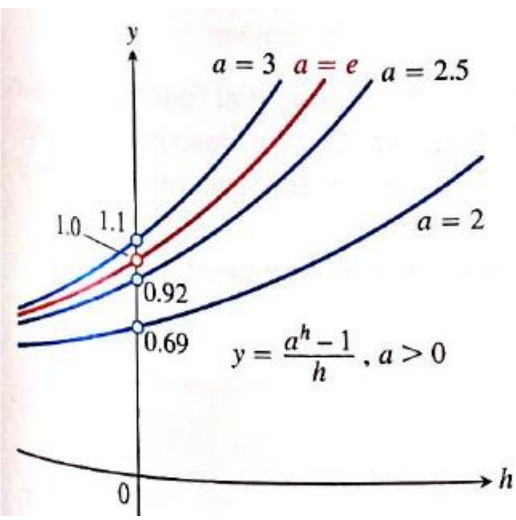
Wednesday May 6

Calculus Unit: Transcendental Functions
Lesson 3: The Discovery of e.

Objective: Be able to do this by the end of this lesson.
1. Find digits of a base of an exponential function whose derivative is equal to itself.
2. Find the slope of the function $y = e^x$ at various points.

Introduction to Lesson 3

Yesterday we explored how we apply the definition of the derivative to take the derivative of an exponential function, $y = a^x$, where a is a positive number. It turned out that the derivative of any function of that form is equal to La^x, where L is some constant. But then we wondered if there was a base number, a, such that when we took the derivative, L would equal 1. We saw that when we took base numbers that were in between 2.5 and 3 and took their derivatives, L got closer to 1, as seen in the graph below.



$$\frac{d}{dx}(2^x) \approx (0.69)\,2^x,$$
$$\frac{d}{dx}\left(2.5^x\right) \approx (0.92)\,2.5^x,$$
$$\frac{d}{dx}\left(3^x\right) \approx (1.10)\,3^x.$$

1. Hopefully yesterday, you found that 2.7 got closer to L approaching 1. Try to find one more digit, the digit in the hundredth's place.

It turns out that you can keep testing L values that get closer to 1 and get more and more digits of a very special number in mathematics that we call Euler's number, or e.

The number *e* is the value of *a* that satisfies the equation

$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1, \tag{5}$$

so that

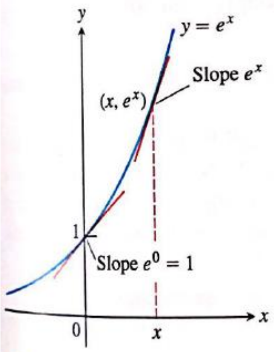
$$\frac{d}{dx}(e^x) = 1 \cdot e^x = e^x. \tag{6}$$

2. This is a big deal! There is a function whose slope at every point is equal to the function evaluated at that same point. Try it out. Graph $y = e^x$ on a graphing calculator or Desmos and find the slope at any point.

To 15 decimal places,

$$e = 2.7\ 1828\ 1828\ 45\ 90\ 45$$

Amazing! You can calculate digits of this number on and on forever and the digits will never repeat. NASA has a website where they've posted the first 2 million digits of e: https://apod.nasa.gov/htmltest/gifcity/e.2mil



6.4 At each x, the slope of the curve $y = e^x$ is e^x.

The Geometric Significance of $\lim_{h \to 0} \frac{a^h - 1}{h}$

You may have noticed that the graph of each of the functions $f(x) = a^x$ passes through the point (0, 1) with a different slope. And what is this slope? It is none other than

$$f'(0) = \lim_{h \to 0} \frac{a^{0+h} - a^0}{h} = \lim_{h \to 0} \frac{a^h - 1}{h}, \qquad (8)$$

the limit at the end of Eq. (1). Thus, Eq. (1) can be reinterpreted to say that if $f(x) = a^x$, then

$$f'(x) = f'(0) \cdot f(x). \qquad (9)$$

The derivative of $y = a^x$ is a^x multiplied by the slope at the point where the graph crosses the y-axis. The graph of e^x crosses the y-axis with slope 1, so e^x is its own derivative (Fig. 6.4).

3. Use the guide above to find the slope at the following points.

x = 0:

x = 1:

x = 2:

Now what if we wanted to take the derivative of e^{5x}? We can, and we'd have to use the chain rule as follows:

If u is a differentiable function of x, then

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}.$$

So d/dx $(e^{5x}) = 5\, e^{5x}$.

4. Now try these examples and check your answers on p. 351.

$$\frac{d}{dx}(e^{kx})$$

$$\frac{d}{dx}(e^{-x})$$

$$\frac{d}{dx}(e^{x^2})$$

$$\frac{d}{dx}e^{\sin x}$$

Thursday, April 30

Calculus Unit: Transcendental Functions
Lesson 4: Practice Working with Exponential Functions and Derivatives of e.

Objective: Be able to do this by the end of this lesson.
1. Sketch shifted exponential curves with a common HASY.
2. Use the chain rule to find derivatives of exponential functions.

Introduction to Lesson 4
Today we're going to do some practice problems to help you better understand exponential functions and their derivatives.

On a separate sheet of paper, do the following Exercises on p. 353:

7, 9, 27, 29, 31, 33, 35, 39, 41

Friday, May 1

Calculus Unit: Applications of Integrals
Lesson 5: Minor Assessment

Objective: Be able to do this by the end of this lesson.
1. Demonstrate mastery of the laws of exponents and taking derivatives of exponential functions.

Calculus
May 4 – May 8

Calculus – Minor Assessment

Name: ___________________________

When you finish, get out your red pen and correct any incorrect answers.
Use the laws of exponents to simplify the following expressions.

1) $2^{\sqrt{3}} \cdot 7^{\sqrt{3}}$

2) $16^{2} \cdot 16^{-1.75}$

3) $\left(\frac{2}{\sqrt{2}}\right)^{4}$

Take the derivative.

4) $y = e^{3x^2}$

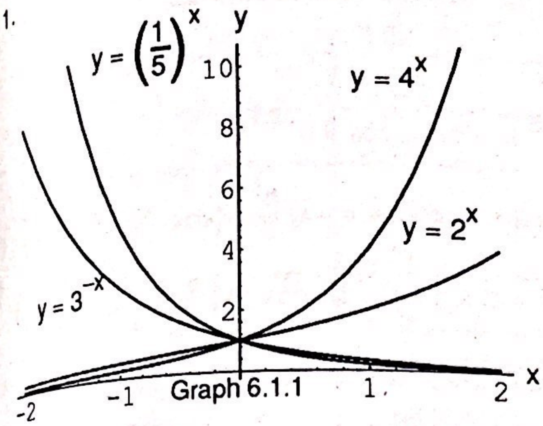
5) $y = 2e^{tan\theta}$

6) $y = 2te^{\sqrt{t}}$

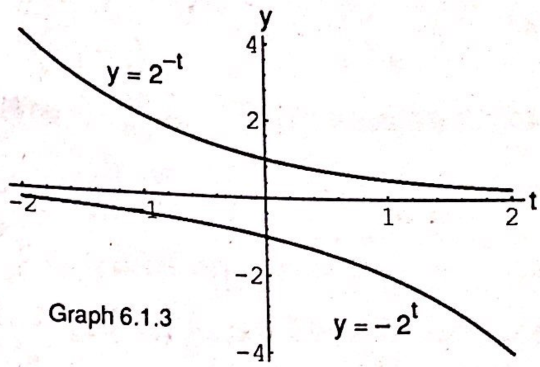
7) $y = xe^{x} - e^{x}$

6.1 EXPONENTIAL FUNCTIONS AND THE DERIVATIVE OF e^x

1.



3.



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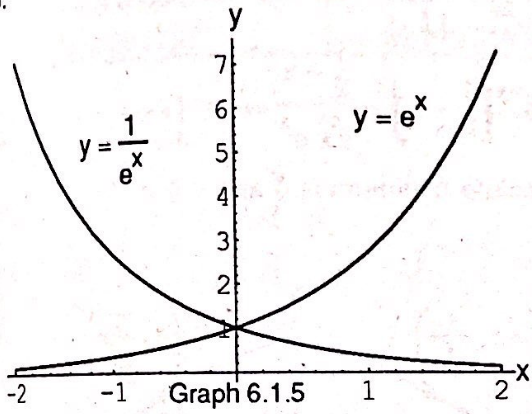

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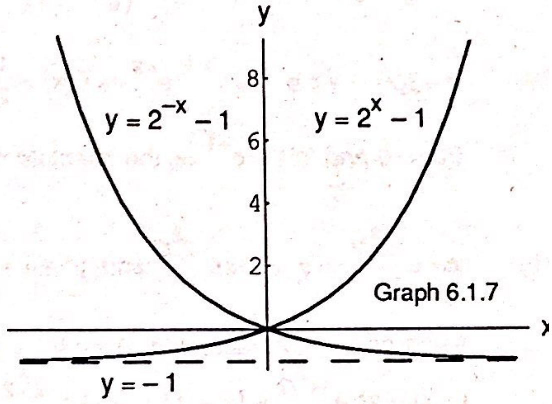
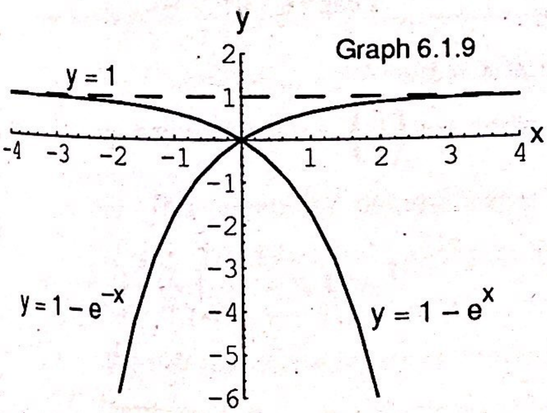


9.



11. a) $\lim_{x \to \infty} (1.01)^x = \infty$ b) $\lim_{x \to -\infty} (1.01)^x = 0$

13. $\lim_{x \to \infty} \left(4 + e^{-x}\right) = 4 + 0 = 4$

15. $\lim_{t \to \infty} \dfrac{3}{2 + 3e^{-t}} = \dfrac{3}{2 + 0} = \dfrac{3}{2}$

17. $16^2 \cdot 16^{-1.75} = 16^{0.25} = 16^{1/4} = 2$

19. $\dfrac{4^{4.2}}{4^{3.7}} = 4^{0.5} = 4^{1/2} = 2$

21. $\left(25^{1/8}\right)^4 = 25^{1/2} = 5$

23. $2^{\sqrt{3}} \cdot 7^{\sqrt{3}} = (2 \cdot 7)^{\sqrt{3}} = 14^{\sqrt{3}}$

25. $\left(\dfrac{2}{\sqrt{2}}\right)^4 = \left(\sqrt{2}\right)^4 = \left(2^{1/2}\right)^4 = 2^2 = 4$

27. $y = e^{-5x} \Rightarrow y' = -5e^{-5x} = \dfrac{-5}{e^{5x}}$

29. $y = e^{2x/3} \Rightarrow y' = \dfrac{2}{3} e^{2x/3}$

31. $y = e^{x + \sqrt{2}} \Rightarrow y' = e^{x + \sqrt{2}}$

33. $y = e^{3x^2} \Rightarrow y' = 6xe^{3x^2}$

35. $y = e^{5t^2 - 7t} \Rightarrow \frac{dy}{dt} = (10t - 7)\left(e^{5t^2 - 7t}\right)$

37. $y = 2e^{\tan\theta} \Rightarrow \frac{dy}{d\theta} = 2\sec^2\theta\, e^{\tan\theta}$

39. $y = x e^x \Rightarrow y' = e^x + x e^x$

41. $y = x e^x - e^x \Rightarrow y' = x e^x + e^x - e^x = x e^x$

43. $y = \left(6x^2 + 6x + 3\right) e^{2x} \Rightarrow y' = (12x + 6)\, e^{2x} + \left(6x^2 + 6x + 3\right)(2)\, e^{2x} = 12\, e^{2x}\left(x^2 + 2x + 1\right)$

45. $y = 2t\, e^{\sqrt{t}} = 2t\, e^{t^{1/2}} \Rightarrow \frac{dy}{dt} = 2\, e^{t^{1/2}} + (2t)\left(\frac{1}{2}t^{-1/2}\right) e^{t^{1/2}} = \left(2 + t^{1/2}\right) e^{t^{1/2}} = \left(\sqrt{t} + 2\right) e^{\sqrt{t}}$

47. $y = \frac{e^x}{e^{-x} + 1} \Rightarrow y' = \frac{e^x\left(e^{-x} + 1\right) - e^x\left(-e^{-x}\right)}{\left(e^{-x} + 1\right)^2} = \frac{1 + e^x + 1}{\left(e^{-x} + 1\right)^2} = \frac{e^x + 2}{\left(e^{-x} + 1\right)^2}$

49. $y = (\cos\theta - 1)\, e^{\cos\theta} \Rightarrow \frac{dy}{d\theta} = (-\sin\theta)\, e^{\cos\theta} + (\cos\theta - 1)(-\sin\theta)\, e^{\cos\theta} = -\sin\theta\cos\theta\, e^{\cos\theta}$

51. $y = \frac{e^t}{2}(\sin t + \cos t) \Rightarrow \frac{dy}{dt} = \frac{e^t}{2}(\sin t + \cos t) + \frac{e^t}{2}(\cos t - \sin t) = (\sin t + \cos t + \cos t - \sin t)\frac{e^t}{2} = (\cos t)\, e^t$

53. $y = \frac{e^{-x}}{5}(2\sin 2x - \cos 2x) \Rightarrow y' = -\frac{e^{-x}}{5}(2\sin 2x - \cos 2x) + \frac{e^{-x}}{5}(4\cos 2x + 2\sin 2x) = (5\cos 2x)\frac{e^{-x}}{5} = (\cos 2x)\, e^{-x}$

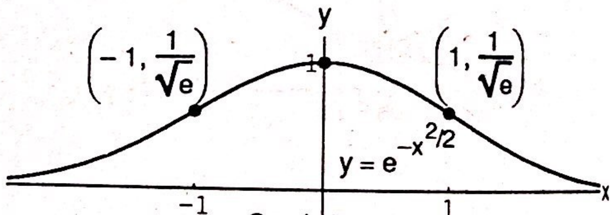
55. $y = \frac{ax - 1}{a^2} e^{ax} \Rightarrow y' = \frac{1}{a}\left(e^{ax}\right) + \left(\frac{ax - 1}{a^2}\right)(a)e^{ax} = \left(\frac{1}{a} + x - \frac{1}{a}\right)e^{ax} = x\, e^{ax}$

57. $y = \frac{e^{2\theta}}{e^{2\theta} + 1} \Rightarrow \frac{dy}{d\theta} = \frac{2e^{2\theta}\left(e^{2\theta} + 1\right) - e^{2\theta}\left(2e^{2\theta}\right)}{\left(e^{2\theta} + 1\right)^2} = \frac{2e^{2\theta}\left(e^{2\theta} + 1 - e^{2\theta}\right)}{\left(e^{2\theta} + 1\right)^2} = \frac{2\, e^{2\theta}}{\left(e^{2\theta} + 1\right)^2}$

59. $y = f(x) = \sqrt{x}\, e^{-x} = x^{1/2}\, e^{-x} \Rightarrow f'(x) = \frac{1}{2}x^{-1/2}e^{-x} + x^{1/2}(-1)\, e^{-x} = x^{-1/2}\, e^{-x}\left[\frac{1}{2} - x\right] = \frac{\frac{1}{2} - x}{\sqrt{x}\, e^x} \Rightarrow f'\ \underset{0}{[}+++\underset{1/2}{|}---\underset{1}{]}$,

$f(0) = 0$ and $f(1) = e^{-1} \Rightarrow$ the absolute maximum is $\frac{1}{\sqrt{2e}}$ at $x = 1/2$, the absolute minimum is 0 at $x = 0$.

61. $y = e^{-x^2/2} \Rightarrow y' = -x e^{-x^2/2}$ and $y'\ +++\underset{0}{|}--- \Rightarrow$ the curve is rising on $(-\infty, 0)$, falling on $(0, \infty)$; $y'' = (-1)e^{-x^2/2} + (-x)(-x)\, e^{-x^2/2} = (x + 1)(x - 1)\, e^{-x^2/2}$ and $y''\ +++\underset{-1}{|}---\underset{1}{|}+++ \Rightarrow$ the curve is concave upward on $(-\infty, -1)$ and $(1, \infty)$, concave downward on $(-1, 1) \Rightarrow$ points of inflection at $x = \pm 1$



Graph 6.1.61

63. $y = 2\, e^{\sin(x/2)} \Rightarrow y' = 2\left(\cos\left(\frac{x}{2}\right)\right)\left(\frac{1}{2}\right) e^{\sin(x/2)} = \left(\cos\left(\frac{x}{2}\right)\right) e^{\sin(x/2)} \Rightarrow$ when $\cos\left(\frac{x}{2}\right) = 0$ the extreme occur i.e. at the odd multiples of π; the maximum is $2e$, the minimum is $2/e$

65. $f(x) = e^{x/(x^2 + 1)} \Rightarrow f'(x) = \frac{1\left(x^2 + 1\right) - x(2x)}{\left(x^2 + 1\right)^2} e^{x/(x^2 + 1)} = \frac{(1 - x)(1 + x)}{\left(x^2 + 1\right)^2} e^{x/(x^2 + 1)}$ and $f'\ ---\underset{-1}{|}+++\underset{1}{|}--- \Rightarrow$ the absolute maximum is $\sqrt{e}$ at $x = 1$, the absolute minimum is $1/\sqrt{e}$ at $x = -1$

67. $h(t) = e^{t\sqrt{4 - t^2}} \Rightarrow h'(t) = \left(\sqrt{4 - t^2} + t\left(\frac{1}{2}\right)\left(4 - t^2\right)^{-1/2}(-2t)\right) e^{t\sqrt{4 - t^2}} = \frac{2\left(\sqrt{2} - t\right)\left(\sqrt{2} + t\right)}{\sqrt{4 - t^2}} e^{t\sqrt{4 - t^2}}$,

$h'\ \underset{-2}{[}---\underset{-\sqrt{2}}{|}+++\underset{\sqrt{2}}{|}---\underset{2}{]}$, $h(-2) = 1$, $h(-\sqrt{2}) = e^{-2}$, $h(\sqrt{2}) = e^2$ and $h(2) = 1 \Rightarrow$ the absolute maximum is e^2 at $t = \sqrt{2}$, the absolute minimum is $e^{-2} = 1/e^2$ at $t = -\sqrt{2}$

69. $\frac{dy}{dt} = ky$

71. $\frac{dp}{dt} = kp$

73. $f(x) = e^x \Rightarrow f'(x) = e^x \Rightarrow L(x)$ at $x = 0$ is $f'(0)(x - 0) + f(0) = 1(x) + 1 \Rightarrow L(x) = x + 1$

Solutions to Minor Assessment

1) $14^{\sqrt{3}}$

2) 2

3) 4

4) $y' = 6xe^{3x^2}$

5) $y = 2sec^2\theta \cdot e^{tan\theta}$

6) $\left(\sqrt{t} + 2\right)e^{2t}$

7) xe^x