Calculus I

April 6-April 9

Time Allotment: 40 minutes per day

Student Name: ________________________________

Teacher Name: ________________________________

Calculus
April 6-9

Packet Overview

Date	Objective(s)	Page Number
Monday, April 6	1. Definite Integral Review 2. Find Area Between Two Curves	2-6
Tuesday, April 7	1. Practice Finding Area Between Two Curves	7-11
Wednesday, April 8	1. Finding Area Between Three or More Functions	12-16
Thursday, April 9	1. Finding Horizontal Areas Between Curves	17-23
Friday, April 10	Off	

Additional Notes: Dear Students: you now have the calculus skills in hand to do some incredible things this week, things that you definitely did not think possible last year, and maybe didn't think possible last week. I really hope you are enjoying these lessons on integrals, and while we are not able to share them as a class right now, I hope that you might be able to share even some small beauty you have discovered with your family and those around you.

As usual, be on the lookout for graphs that need to be graphed by you, and for questions you need to answer. Most places for you to answer questions will be inside boxes I drew, but there may be a few questions you need to answer without a box or problems I ask you to work on scratch paper. Be sure as always to work all assigned example problems and exercises. For the exercises and quizzes, do your best to work the problem with a pencil, then check the solutions on the last pages of your packet with a red pen. Ask if you have questions!

Though not required to complete these assignments, Khan Academy's AP Calculus AB series of videos are a helpful resource for supplemental learning. We'll be going over the chapter on Applications of Integrals

Academic Honesty

I certify that I completed this assignment independently in accordance with the GHNO Academy Honor Code.

Student signature:

____

I certify that my student completed this assignment independently in accordance with the GHNO Academy Honor Code.

Parent signature:

____

Monday, April 6

Calculus Unit: Integral Applications
Lesson 1: Finding the Area Between Two Functions

Unit Overview: Integral Applications
Once more, all of our hard work learning the theory and mechanics behind integrals is about to pay off! Our next unit will explore applications of integrals. Some of these applications include applying integrals to quantities in physics (position, velocity, acceleration, forces, work, energy, etc.) and calculating areas and volumes of geometric shapes, including deriving the volume formulas of cylinders, cones, and spheres. By the end of this unit, you will understand more deeply why the physics equations we use in physics class work. You will also never have to memorize another physics equation for the rest of physics I, nor will you have to remember area and volume equations because you will be able to use calculus to derive them!

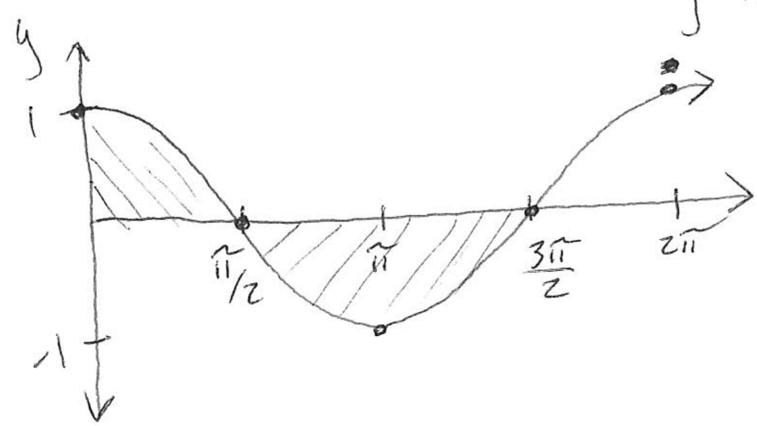
Objective:
1. Review graphical representation of definite integrals.
2. Apply integral properties to pinpoint the area we want to find in between the curves of two functions.

Introduction to Lesson 1
Today we're going to review what definite integrals graphically show us. We'll start by finding the area between the curve and x-axis of a regular old trig function, with an emphasis on remembering that the area above the curve is positive, the area below the curve is negative, and the total area we get from any definite integral is the net sum of any positive and negative area. Then we will explore how it is possible, given *two* functions, to find the area *between* two curves. The question to ponder is how can we do that?

Lesson 1 - 1

Before we start something new, let's review definite integrals applied to trig functions.

$y = f(x) = \cos(x)$



$$\int_0^{\pi/2} \cos x \, dx$$

$$= \sin x \Big|_0^{\pi/2}$$

$$= \sin \tfrac{\pi}{2} - \sin 0 = 1$$

Here, we're finding the area under the curve of the cosine function from zero to $\pi/2$.

Now let's find the area between $\pi/2$ and $3\pi/2$. Can you look at the graph and guess what it might be? Set up your definite integral below and check your guess.

a)

Now let's go from 0 to $3\pi/2$. Guess, then set up the integral.

b)

Lastly, do 0 to 2π. You don't have to set up the integral.

c)

Notice how each time, we're finding the net area above the x-axis. Any area below the x-axis gives us "negative area."

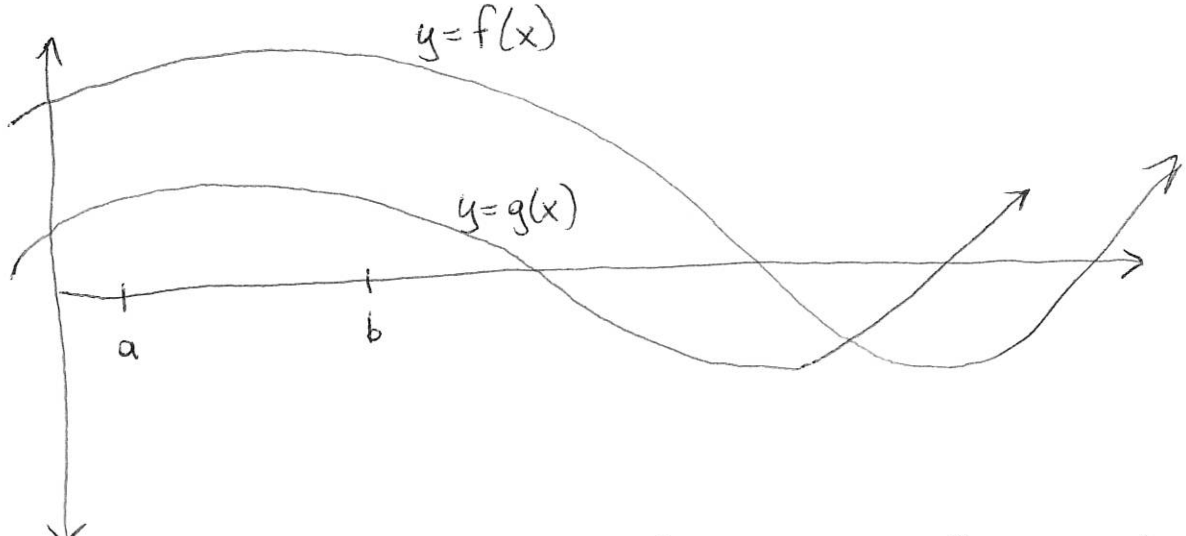
Answers: a) $\int_{\pi/2}^{3\pi/2} \cos x \, dx = -2$ b) $\int_0^{3\pi/2} \cos x \, dx = -1$ c) 0

Lesson 1 - 2

So far, we've only considered finding the area between a function and the x-axis. Can you think of some other areas of graphs we could use calculus to determine? Write a few below.

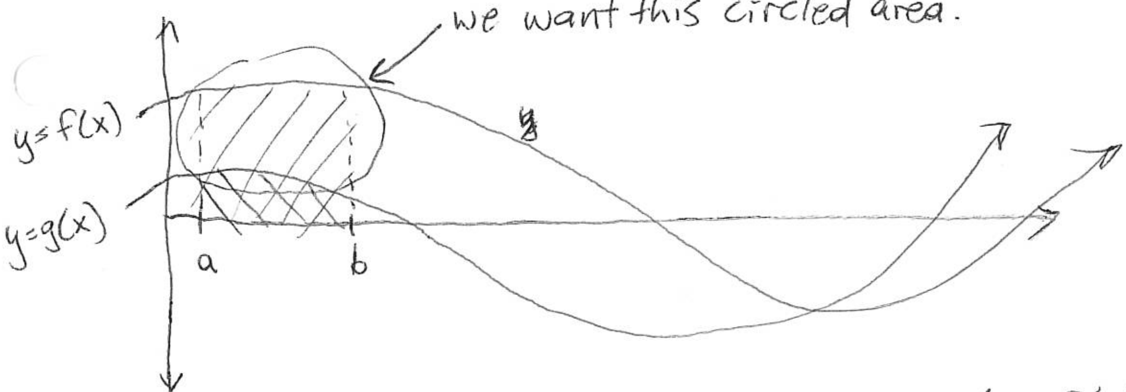
What we're going to do next is find out how to ~~fi~~ find the area between two curves. Before you go on, write 1-2 complete sentences below describing how you might be able to do this.

Draw ~~on~~ the graph below to illustrate your point before going on to the next page. How can definite integrals help us find the area from a to b?



Write a definite integral to find the area from a to b.

Lesson 1 - 3

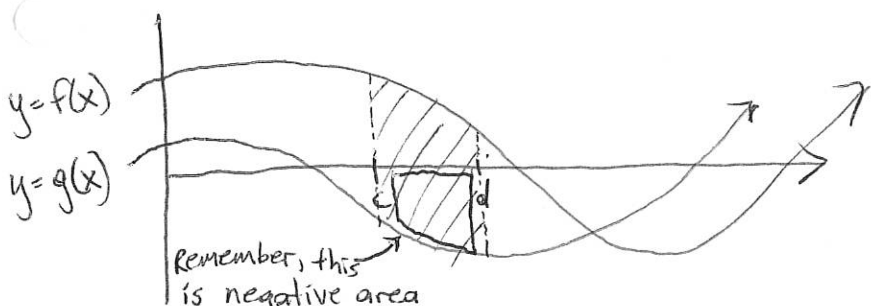


To do this, we want to find the area under f(x), then subtract the area under g(x).

The definite integral will look like this:

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x) - g(x))\,dx$$

Let's look at another interval:



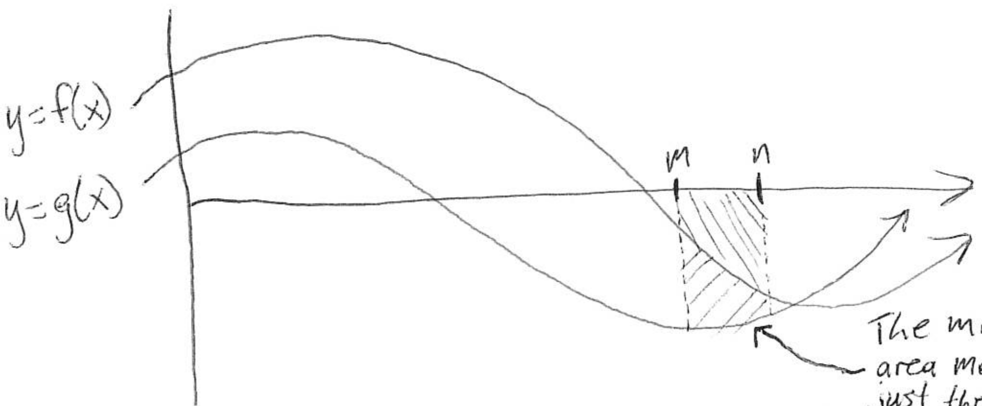
Write your guess below for ~~the~~ what the definite integral will be then turn the page to see if you got it!

Lesson 1 - 4

$$\int_c^d f(x)\,dx - \int_c^d g(x)\,dx = \int_c^d \big(f(x) - g(x)\big)\,dx$$

Notice how the minus sign makes the negative area positive, and gives us the entire net area.

Finally, how can we get the area between two curves below the x-axis. By now, you can probably guess how: the same way as we've done for the other cases



The minus sign subtracting off the g(x) area makes that area positive, leaving just the area between the two curves.

$$\int_m^n f(x)\,dx - \int_m^n g(x)\,dx = \int_m^n \big(f(x) - g(x)\big)\,dx$$

Tomorrow, we'll look at an example proplem

Tuesday, April 7

Calculus Unit: Finding Area Between Curves Practice
Lesson 2: Connecting Position, Velocity, and Acceleration Functions Using Integrals

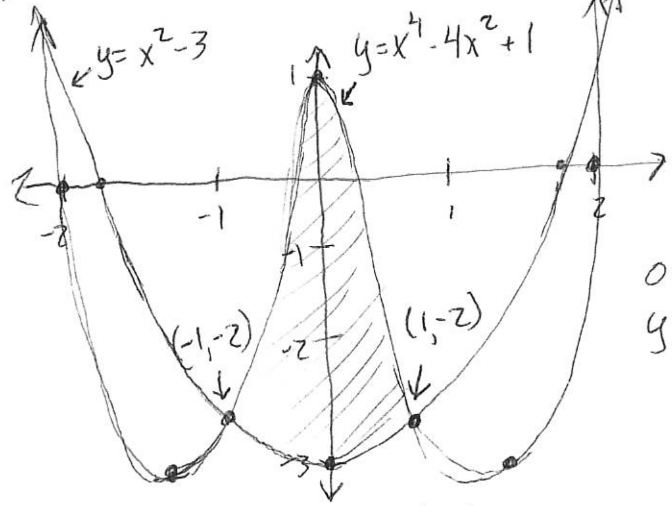
Objective: Be able to do this by the end of this lesson.
1. Practice Solving Problems Asking to Find Area Between Two Curves.
2. Find intersection points to determine bounds of definite integral.
3. Sketch graphs of curves to verify you are subtracting the correct function from the other.
4. Determine whether the area you have found is reasonable given the graph.

Introduction to Lesson 2
Now we're going to practice finding the area between functions. You'll be given a graph for some exercises, and some you will just be given the two functions. In the latter case, always sketch a graph of the functions before you start finding intersection points or set up the integral.

Lesson 2 - 1

Let's do an example problem and then let you try some practice exercises on your own!

Ex:



Here's a sketch of two functions,

$$y = x^4 - 4x + 1$$

$$y = x^2 - 3$$

I would recommend graphing them on your calculator or Desmos so you can see it better.

Our goal is to find the area of the shaded region.

First, let's find our lower and upper bounds (or you can call them the left and right-most boundaries). These will be the intersection points. If you plug in -1 to both functions, you will get -2. If you plug in 1 to both functions, you will get -2, so these are our intersection points. -1 is our lower bound, and 1 is our upper bound. $\int_{-1}^{1} (\quad) dx$

Second, we need to fill in the parentheses. To do this, we'll take the upper function, $y = x^4 - 4x + 1$ and subtract the lower function, $y = x^2 - 3$.

$$\int_{-1}^{1} \left(x^4 - 4x^2 + 1 - (x^2 - 3)\right) dx$$

Try working this integral on a separate sheet of scratch paper. Then check your answer.

$$= \int_{-1}^{1} (x^4 - 5x^2 + 4)\, dx$$

$$= \frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x \Big|_{-1}^{1}$$

$$= \frac{1}{5} - \frac{5}{3} + 4 - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)$$

$$= \frac{2}{5} - \frac{10}{3} + 8 = 8 + \frac{6}{15} - \frac{50}{15}$$

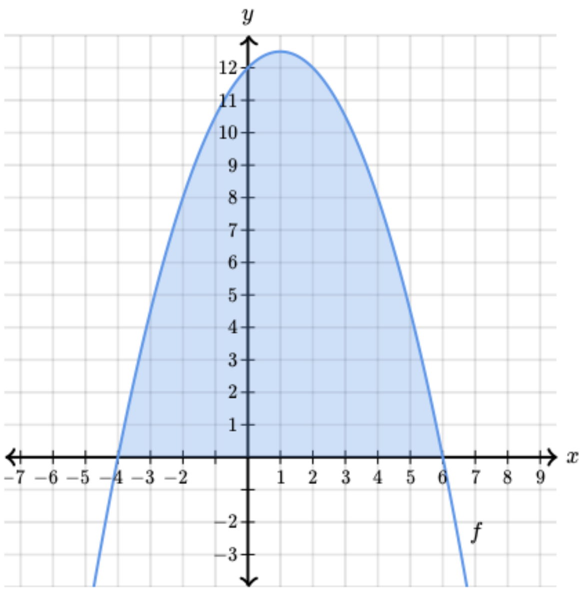
$$\boxed{= 8 - \frac{44}{15} = \frac{76}{15}}$$

Calculus I – Lesson 2 Exercises : Finding Areas Under, Above, and Between Curves

1)

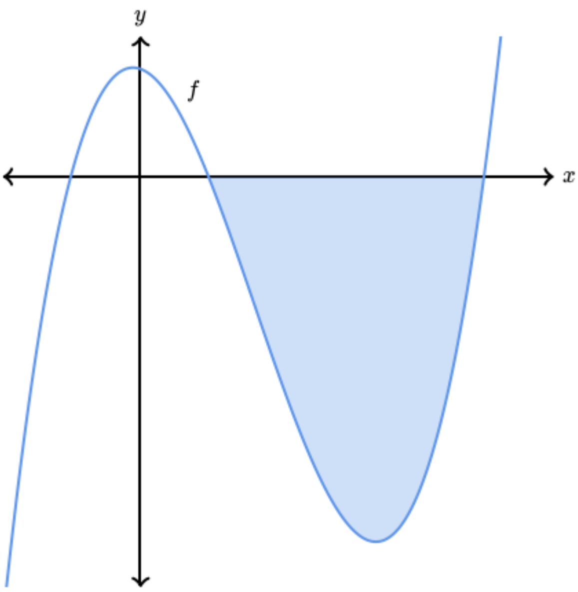
$f(x) = 12 + x - \frac{1}{2}x^2$

and the x-axis.



What is its area?

2) The shaded region is bounded by the graph of the function $f(x) = x^3 - 5x^2 - x + 5$ and the x-axis.

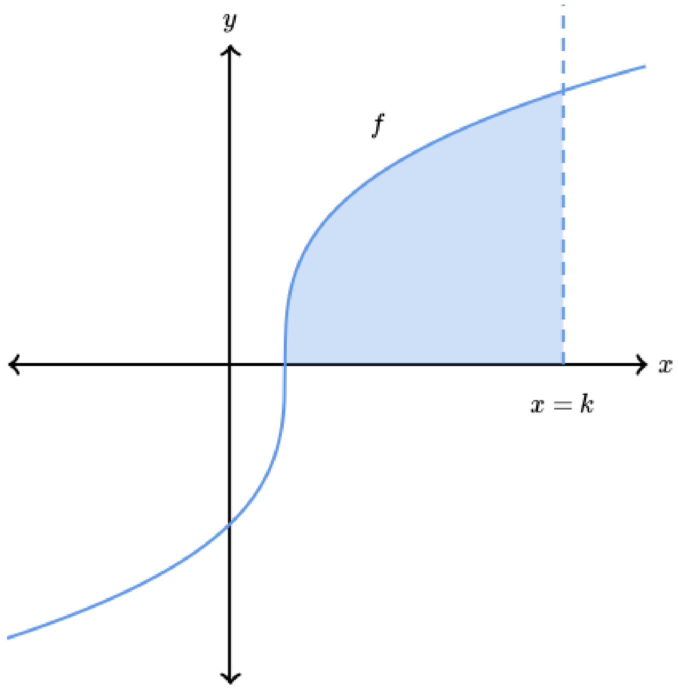


What are the zeroes of this function?

Find the shaded area if the zeros of the function are -1, 1, and 5.

3)

The shaded region is bounded by the graph of the function $f(x) = \sqrt[3]{x-1}$, the line $x = k$, and the x-axis.



If the region has area 12, what is the *exact* value of k?

4)

What is the area of the region between the graphs of $f(x) = 2x^2 + 5x$ and $g(x) = -x^2 - 6x + 4$ from $x = -4$ to $x = 0$?

5)

What is the area of the region between the graphs of $f(x) = \sqrt{x+10}$ and $g(x) = x - 2$ from $x = -10$ to $x = 6$?

6)

What is the area of the region between the graphs of $f(x) = x^2 - 3x$ and $g(x) = 2x$ from $x = 0$ to $x = 5$?

Wednesday, April 8

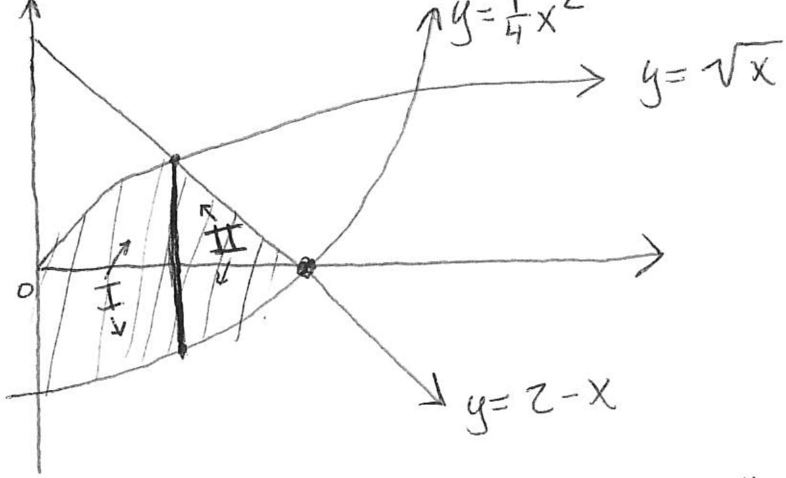
Calculus Unit: Applications of Integrals
Lesson 3: Composite Area Between Curves

Objective: Be able to do this by the end of this lesson.
1. Use same integral techniques as before to find area when 3 or more functions bound an area.
2. Use integral properties we've learned to segment an area, then add the areas together.

Introduction to Lesson 3
You'll learn in this section how to find an area between functions when there are more than two functions surrounding a given area. Again, some exercises will give you a graph; for others you will have to sketch your own.

Lesson 3 - 1 Composite Area Between Curves

Now we've got 3 functions and want to find the area between them. Let's find out how to do it!



In order to do it, we need to divide the shaded region above into two separate regions, then add their areas.

What is the point of intersection of $y=\sqrt{x}$ and $y=2-x$?

a)

Region I: $\int_0^1\left(\sqrt{x}-\left(\frac{x^2}{4}-1\right)\right)dx$

Region II: $\int_1^2\left(2-x-\left(\frac{x^2}{4}-1\right)\right)dx$

Combined: $\int_0^1\left(\sqrt{x}-\frac{x^2}{4}+1\right)dx+\int_1^2\left(3-x-\frac{x^2}{4}\right)dx$ Try this on your own on a separate sheet and check your answer.

$$= \frac{2}{3}x^{3/2}-\frac{x^3}{12}+x\Big|_0^1 + 3x-\frac{x^2}{2}-\frac{x^3}{12}\Big|_1^2$$

$$= \frac{2}{3}-\frac{1}{12}+1 \quad + \quad 6-2-\frac{8}{12}-\left(3-\frac{1}{2}-\frac{1}{12}\right)$$

$$= \frac{19}{12} + \frac{48}{12}-\frac{8}{12}-\frac{36}{12}+\frac{6}{12}+\frac{1}{12}$$

$$\boxed{=\frac{30}{12}=\frac{5}{2}}$$

Answers: a) (1,1)

Calculus I – Lesson 3: Finding the Composite Area Between Three or More Functions

1)

What is the area of the region enclosed by the graphs of $f(x) = x^2 + 2x + 11$, $g(x) = -4x + 2$, and $x = 0$?

2)

The graphs of the functions $f(x) = \sin(x)$ and $g(x) = \frac{1}{2}$ intersect at 2 points on the interval $0 < x < \pi$.

3)

What is the area of the region bound by the graphs of $f(x) = -x^2$, $g(x) = 3x - 10$, and $x = 0$ in quadrant IV ?

4)

What is the area of the region between 2 consecutive points where the graphs of $f(x) = \cos(x)$ and $g(x) = -\cos(x) + 2$ intersect?

5)

What is the area of the region bound by the graphs of $f(x) = \sqrt{x-2}$, $g(x) = 14 - x$, and $x = 2$?

Thursday, April 9

Calculus Unit: Applications of Integrals
Lesson 4: Horizontal Areas Between Curves

Objective: Be able to do this by the end of this lesson.
1. Find the area under a curve if the curve is on the y-axis.

Introduction to Lesson 4
You've gotten pretty good at finding curves above and below the x-axis. Do you think it's possible to find areas to the left and right of the y-axis then?! We'll explore that possibility in this lesson. Get excited! Don't forget you can always rotate the axes so that your infinitesimally narrow horizontal rectangles appear their usual vertical orientation. Just keep the labelling of the axes consistent.

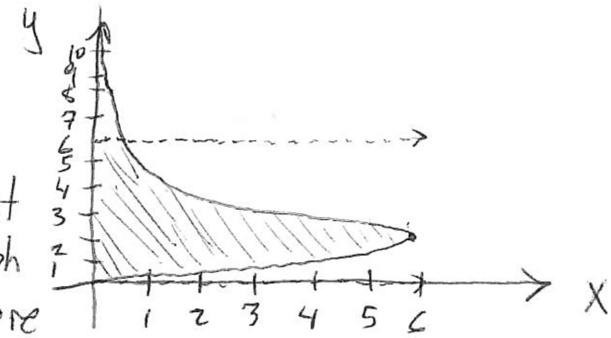
Lesson 4 - Horizontal Areas Between Curves - 1

We've been integrating sideways, from left-to-right.
Now let's see how to integrate up and down the y-axis!

Take this function: $x = \dfrac{144y}{(y^2+4)^2}$ and find the area bounded by the y-axis, $y=0$, and $y=6$.

Follow along 1 step at a time on a separate sheet of paper. Pretend I'm at the board and you're taking notes, working the steps for yourself.

First let's graph the function:



Setting up the problem actually isn't that bad. All you have to do is rotate the graph 90° and use y instead of x when we're integrating.

$$\int_0^6 \left(\frac{144y}{(y^2+4)^2}\right) dy$$

This would be a great opportunity to practice your u-sub skills, so please try to work this integral on your own before you look at my steps.

$$144\int_0^6 \frac{y}{(y^2+4)^2}\, dy$$

pull out the constant and don't panic that you're going to u-sub in the denominator. It will work out

$$\boxed{u} = \boxed{y^2+4}$$
$$du = 2y\, dy$$
$$dy = \boxed{\frac{du}{2y}}$$

sub in u for y^2+4 and $\frac{du}{2y}$ for dy.

$$\cancel{144}^{\,72}\int_4^{40} \frac{\cancel{y}}{u^2}\,\frac{du}{\cancel{2y}}$$

cancel y's, divide 144 by 2, and change your bounds: $u = y^2+4 \Rightarrow u_{lower} = 0^2+4 = 4$
$u_{upper} = 6^2+4 = 40$

You don't have to change bounds in this step. You can use the other method we discussed where you sub y back in for u at the end, but for this integral, I think this method is easier.

$$= 72\int_4^{40} u^{-2}\, du$$

Integrate that integral then flip the page!

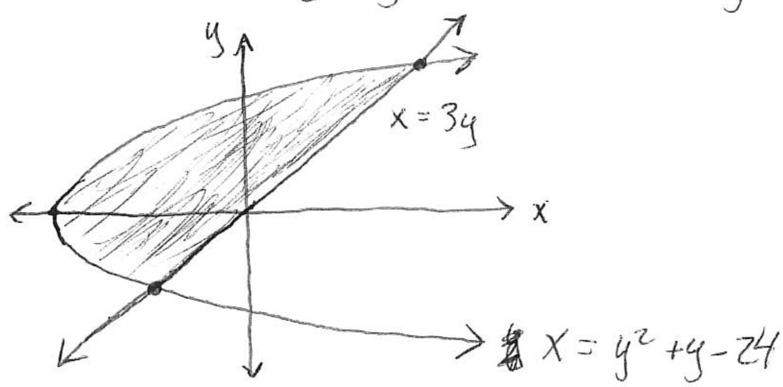
Lesson 4 - 2

$$72\frac{u^{-1}}{-1}\Big|_4^{40}$$

$$= -72u^{-1}\Big|_4^{40} = \frac{-72}{40} - \left(-\frac{72}{4}\right) = \boxed{16.2 \text{ units}^2}$$

Great job if you got it. Ask if I can help you.

What if we want to find the horizontal area between two curves?
We can do that! Take $x = y^2 + y - 24$ and $x = 3y$

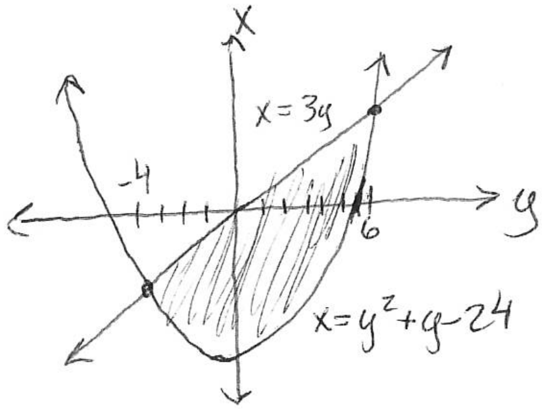


We can integrate bottom to top just like last time, but what's missing?
That's right: the boundaries. How can we find them?
Yep. Find the intersections of the two functions.
Do that below, then flip the page to continue the adventure...

Find where $x = y^2 + y - 24$ and $x = 3y$ intersect

Lesson 4 - 3

Hopefully you got $y=-4$ and $y=6$ as your intersections. They can be used as the boundaries of your definite integral. Since we're integrating from $y=-4$ to $y=6$, we'll need to make $x=3y$ our top function and subtract $x=y^2+y-24$. To see why that's the case, let's rotate the graph 90°.



Now lets take the area of $x=3y$ and subtract off the area of y^2+y-24:

$$\int_{-4}^{6} \left(3y-\left(y^2+y-24\right)\right) dy$$

$$=\int_{-4}^{6} \left(-y^2+2y+24\right) dy$$

Then evaluate and show your steps in the box below:

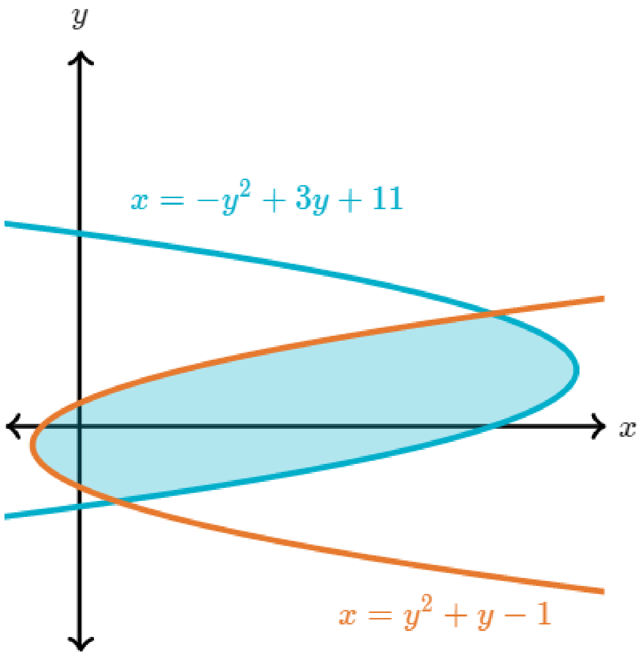
Answer: $\frac{500}{3} \approx 166.67$

Calculus I – Lesson 4 Exercises: Finding the Area Along the y-axis

Given the following functions and graphs, find the area for each exercise.

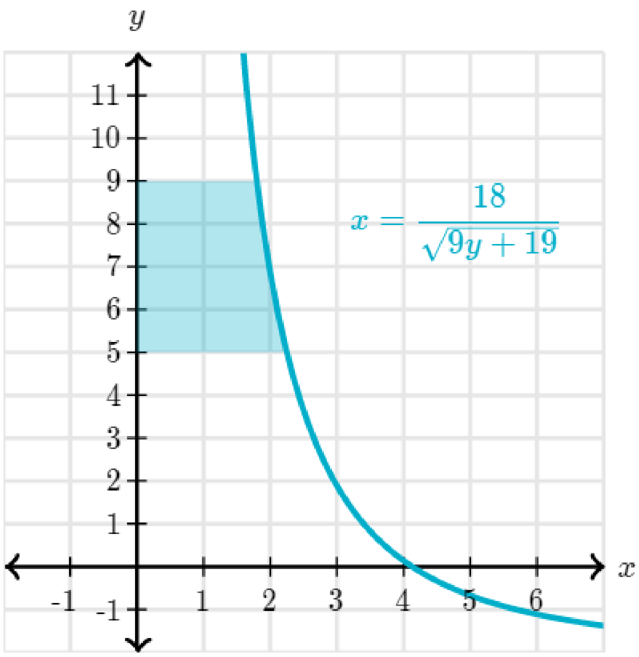
1)

The curves $x = -y^2 + 3y + 11$ and $x = y^2 + y - 1$ are graphed.



2)

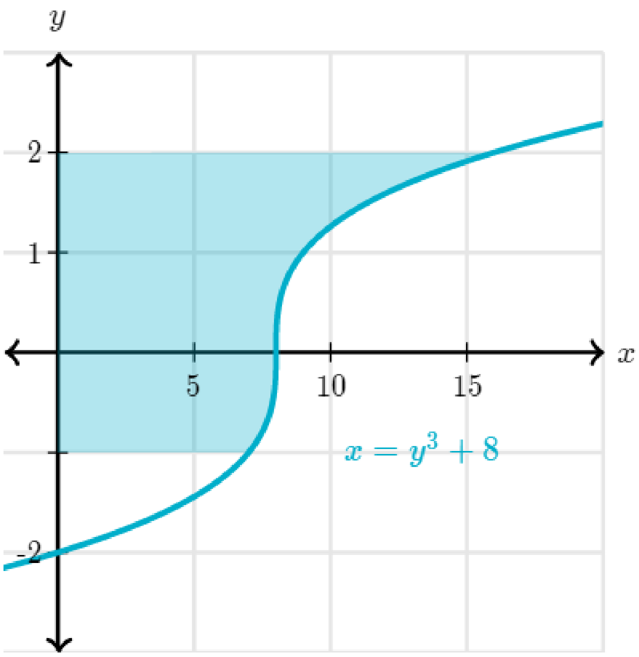
The curve $x = \dfrac{18}{\sqrt{9y+19}}$ is graphed.



What is the area bounded by the curve, the y-axis, the line $y = 5$ and the line $y = 9$?

3)

The curve $x = y^3 + 8$ is graphed.



What is the area bounded by the curve, the y-axis, the line $y = -1$ and the line $y = 2$?

Answer key

Lesson 2

1) $\frac{250}{3}$

2) $\frac{128}{3}$

3) $K = 9$

4) 40 units^2

5) $\frac{320}{3}$

6) $\frac{125}{6}$

Lesson 3 (Graph these if you get stuck)

1) 9

2) Intersection points: $x = \frac{\pi}{6}, \frac{5\pi}{6}$
Area $= \sqrt{3} - \frac{\pi}{3}$

3) Intersection points: $x = 0, 2$
Area $= \frac{34}{3}$

4) Intersection points: $x = 0, 2\pi$
Area $= 4\pi$

5) Intersection points: $x = 2, 11$
Area $= \frac{99}{2}$

Lesson 4

1) $\int_{-2}^{3} (-y^2 + 3y + 11)\,dy - \int_{-2}^{3} (y^2 + y - 1)\,dy$
$= \frac{125}{3} = 41.67$

2) $\int_{5}^{9} \left(\frac{18}{\sqrt{9y+19}}\right) dy = 4\sqrt{9y+19}\,\Big|_5^9 = 8$

3) $\int_{-1}^{2} (y^3 + 8)\,dy = \frac{111}{4}$

Lesson 5

1) $\int_{\frac{\pi}{2}}^{2\pi} \left(3\cos y + \frac{12}{\pi}\right) dy = 15$