GreatHearts

Calculus I

May 11 – May 15

Time Allotment: 40 minutes per day

Student Name: ________________________________

Teacher Name: ________________________________

Packet Overview

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Academic Honesty

I certify that I completed this assignment independently in accordance with the GHNO Academy Honor Code.

Student signature:

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I certify that my student completed this assignment independently in accordance with the GHNO Academy Honor Code.

Parent signature:

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Calculus
May 11 – May 15

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Monday, May 11

Calculus Unit: Transcendental Functions
Lesson 1: Exponential Change

Objective: Be able to do this by the end of this lesson.
1. Write the exponential growth formula.
2. Identify its parts.
3. Use the exponential growth formula to solve problems.

Introduction to Lesson 1
Today we're going to apply Euler's number to solving problems that involve exponential growth and decay. The lesson will follow p. 351-353 in your textbook.

Exponential Change

In many instances in science, some positive quantity increases or decreases at a rate that at any given time t is proportional to the amount that is present at time t. In Section 6.5 we will show that in these instances the amount can be represented by an equation of the form $y = y_0 e^{kt}$, where y_0 is the amount initially present at time $t = 0$ and k is a constant.

The equation

$$y = y_0 e^{kt} \tag{10}$$

is called the **law of exponential change.**

Where does this equation come from? While it's derivation is a Calculus II topic, we can still use it now to show its power of modelling exponential growth and decay. Let's turn to a few example problems.

1. In an ideal environment, the mass *m* of a cell will grow exponentially, at least early on. Nutrients pass quickly through the cell wall, and growth is limited only by the metabolism within the cell, which in turn depends on the mas of participating molecules. If we make the reasonable assumption that, at each instant of time, the cell's growth rate is proportional to the mass that has already been accumulated, then write an equation below in the form of the law of exponential change equation above to model the exponential growth.
2. One model for the way diseases spread assumes that the rate at which the number *y* of infected people changes is proportional to *y* itself. The more infected people there are, the faster the disease will spread. The fewer there are, the slower it will spread. If y_0 is the number of infected people at time t = 0, then what equation can we write to determine the number of people infected at any given time t?

Now take the equation you wrote above and plug in k = 0.3 and 25,000 for number of people initially infected. How many people will be infected in 2 years?

Do problems 69-73 odd on p. 354, check your answers in the back of the book, and have a great Monday!

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Tuesday, May 12

Calculus Unit: Transcendental Functions
Lesson 2: Inverse Functions and Their Derivatives

Objective: Be able to do this by the end of this lesson.
1. State what a one-to-one function is. Be able to sketch an example of one.
2. Find the inverse of a function. Explain the process.
3. Graph the inverse of a function.

Introduction to Lesson 2

One-to-One Functions

As you know, a function is a rule that assigns a value from its range to each point in its domain. Some functions assign the same value to more than one point. The squares of -1 and 1 are both 1, and the sines of $\pi/3$ and $2\pi/3$ are both 1/2. Other functions never assume a given value more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct points is called one-to-one.

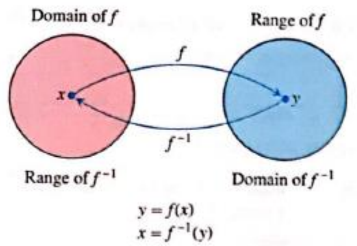
DEFINITION

A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

1. What is a one-to-one function? Write a definition in your own words using complete sentences below. Then write a function and sketch its graph of a one-to-one function.

2. Is y = sin x a one-to-one function? Why or why not? Illustrate your point with a sketch.

3. Write and sketch the graph of a function that is not one-to-one.



6.8 The inverse of a function f sends each output back to the input from which it came.

Inverses

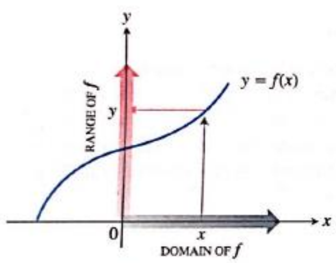
Since each output of a one-to-one function comes from just one input, a one-to-one function can be reversed to send the outputs back to the inputs from which they came. The function defined by reversing a one-to-one function f is called the **inverse** of f. The symbol for the inverse of f is f^{-1}, read "f inverse" (Fig. 6.8). The -1 in f^{-1} is *not* an exponent: $f^{-1}(x)$ does not mean $1/f(x)$.

As Fig. 6.8 suggests, the result of composing f and f^{-1} in either order is the **identity function,** the function that assigns each number to itself. This gives a way to test whether two functions f and g are inverses of one another. Compute $f \circ g$ and $g \circ f$. If $(f \circ g)(x) = (g \circ f)(x) = x$, then f and g are inverses of one another; otherwise they are not. If f cubes every number in its domain, g had better take cube roots or it isn't the inverse of f.

What Functions Have Inverses?

A function has an inverse if and only if it is one-to-one. This means, for example, that increasing functions have inverses and decreasing functions have inverses (Exercises 41 and 42). Functions with positive derivatives have inverses because they increase throughout their domains (Corollary 3 of the Mean Value Theorem, Section 4.6). Similarly, because they decrease throughout their domains, functions with negative derivatives have inverses.

How is the graph of the inverse of a function related to the graph of the function? If the function is increasing, say, its graph rises from left to right, like the graph in Fig. 6.9(a). To read the graph, we start at the point x on the x-axis, go up to the graph, and then move over to the y-axis to read the value of y. If we start with y and want to find the x from which it came, we reverse the process (Fig. 6.9b).



(a) To find the value of f at x, we start at x and go up to the curve and over to the y-axis.

The graph of f is already the graph of f^{-1}, although the latter graph is not drawn in the usual way with the domain axis horizontal and the range axis vertical. The input–output pairs are reversed. To display the graph in the usual way, we have to reverse the pairs by reflecting the graph in the 45° line $y = x$ (Fig. 6.9c) and interchanging the letters x and y (Fig. 6.9d). This puts the independent variable, now called x, on the horizontal axis and the dependent variable, now called y, on the vertical axis.

Notice that the graphs of f and f^{-1} are symmetric about the line $y = x$. This is to be expected because the input–output pairs (a, b) of f have been reversed to produce the input–output pairs (b, a) of f^{-1}. The points (a, b) and (b, a) are symmetric about the line $y = x$.

The pictures in Fig. 6.9 tell us how to express f^{-1} as a function of x, which is stated at the left.

4. Explain in 1 to 2 complete sentences what an inverse function is.

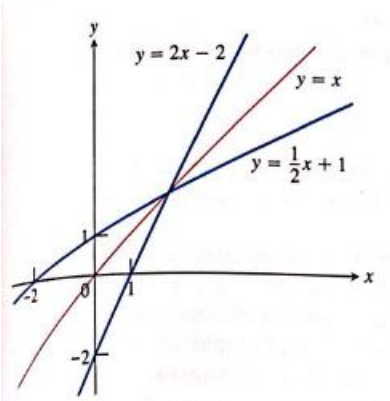
5. Explain the purpose of an identity function. Give an example of one to support your point.

6. Does a function that is not one-to-one have an inverse? Why or why not?

7. Explain in complete sentences how to graph the inverse of a function.

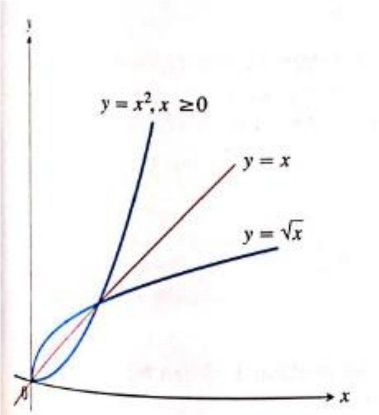
8.

Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.



6.10 Graphing the functions $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$.

9. Find the inverse of the function $y = x^2$, $x \geq 0$, expressed as a function of x.



6.11 The functions $y = \sqrt{x}$ and $y = x^2$, $x \geq 0$, are inverses of one another.

10. In your own words, what does the derivative rule for inverses mean?

11. Work Examples 5 and 6 on p. 359. Then do 1-11 odd on p. 360.

Wednesday May 13

Calculus Unit: Transcendental Functions
Lesson 3: Review of Logarithms

Objective: Be able to do this by the end of this lesson.
1. State the inverse of $y = e^x$.
2. Evaluate expressions using the properties of natural logarithms.

Introduction to Lesson 3

The inverse of the exponential function e^x is called the natural logarithm function, $y = \ln x$. In this section, we study the natural logarithm both as a differentiable function and as a device for simplifying calculations. As we will learn in the process, every exponential function a^x is a numerical power of e^x and every logarithmic function is a numerical multiple of $\ln x$. Thus, we can learn nearly everything we need to know about exponentials and logarithms by studying the functions $y = e^x$ and $y = \ln x$.

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1. What is the inverse of $y = e^x$?

2.

The Natural Logarithm

Since $f(x) = e^x$ is one-to-one, we know that it has an inverse. We call this inverse the natural logarithm function.

DEFINITION

The **natural logarithm function** $y = \ln x$ is the inverse of the exponential function $y = e^x$.

We obtain the graph of $y = \ln x$ by reflecting the graph of $y = e^x$ across the line $y = x$ (Fig. 6.16). The domain of $\ln x$ (range of e^x) is the set of positive real numbers. The range of $\ln x$ (domain of e^x) is the set of all real numbers. The graph of $\ln x$ crosses the x-axis at $(1, 0)$, so $\ln 1 = 0$. Also,

$$\lim_{x \to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x \to \infty} \ln x = \infty. \qquad (1)$$

Because e^x and $\ln x$ are inverses of one another, composing them in either order gives the identity function. This gives two useful equations.

The Inverse Equations for e^x and $\ln x$

$e^{\ln x} = x \quad (x > 0) \quad (2)$ $\qquad \ln(e^x) = x \quad (\text{any } x) \quad (3)$

3. Use the equations above to evaluate the following exponentials.

$e^{\ln(x^2+1)} =$

$e^{3\ln 2} =$

$e^{x+\ln 2} =$

$e^{-\ln x} =$

$\ln e^3 =$

$\ln \sqrt{e} =$

$\ln e^{-1/x} =$

$\ln e^{\sin x}$

4.

Solve $\ln y = 3t + 5$ for y.

5.

Solve the equation $e^{2k} = 10$ for k.

6.

Like the base 10 logarithm you may have studied before, the natural logarithm has the following arithmetic properties.

TABLE 6.3
Properties of natural logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:*	$\ln xy = \ln x + \ln y$	
2. *Quotient Rule:*	$\ln \frac{x}{y} = \ln x - \ln y$	
3. *Reciprocal Rule:*	$\ln \frac{1}{y} = -\ln y$	Rule 2 with $x = 1$
4. *Power Rule:*	$\ln x^y = y \ln x$	

7. Use the properties of natural logarithms above to evaluate the following exponentials.

$\ln 6 =$

$\ln 4 - \ln 5 =$

$\ln \frac{1}{8} =$

$\ln 4 + \ln \sin x =$

$\ln \sec x =$

$\ln \sqrt[3]{x+1} =$

8. Do Exercises 1, 5, 11, and 13 on p. 370. Check your answers in the back of the book.

Thursday, May 14

Calculus Unit: Transcendental Functions
Lesson 4: The Derivative of y = ln x.

Objective: Be able to do this by the end of this lesson.
1. Take the derivative of y = ln x.
2. Apply the chain rule to take derivatives of composite natural log functions.

Introduction to Lesson 4
This is the last new topic we'll cover in Calculus I. We've spent a lot of time reviewing exponentials, discovering e, and reviewing inverse functions and logarithms. We'll finish off the year learning how to take the derivative of the natural log function, ln x.

The Derivative of $y = \ln x$

Theorem 1 in Section 6.2 tells us that the natural logarithm function is differentiable because it is the inverse of a differentiable function whose derivative is never zero. Knowing this, we can calculate what the derivative of the logarithm must be:

$$y = \ln x$$

$$e^y = x \qquad \text{Exponentiate both sides.}$$

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x) \qquad \text{Take the derivative of both sides.}$$

$$e^y \frac{dy}{dx} = 1 \qquad \text{Chain Rule: } \frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{e^y} \qquad \text{We can divide by } e^y \text{ because } e^y \text{ is never zero.}$$

$$\frac{dy}{dx} = \frac{1}{x} \qquad \text{Replacing } e^y \text{ by } x.$$

In short,

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

The Chain Rule extends Eq. (4) to a more general form.

If $u > 0$ is a differentiable function of x, then

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}.$$

1. Try taking the derivative of the following function, then check your answer on p. 364:

$$\frac{d}{dx}[\ln(x^2 + 3)] =$$

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the rules in Table 6.3 to simplify the formulas before differentiating. The process, called **logarithmic differentiation,** is illustrated in the next example.

2.

Find dy/dx if $y = \dfrac{(x^2+1)(x+3)^{1/2}}{x-1}$

3. Do Exercises 19, 21, 23, 25, and 41 on p. 370. Check your answers in the back of the book.

Friday, May 15

Calculus Unit: Applications of Integrals
Lesson 5: Minor Assessment

Objective: Be able to do this by the end of this lesson.
1. Demonstrate mastery of the laws of exponents and taking derivatives of exponential functions.

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Calculus – Minor Assessment

Name: ____________________________

Evaluate the following exponentials

1) $e^{3 \ln 2} =$

2) $\ln \sqrt{e} =$

3) Solve for k.
$e^{2k} = 4$

Take the derivative.
4) y = ln 5x

5) y = ln (x +1)

6) $y = ln\frac{2}{x}$

7) $y = \frac{ln\,x}{x}$

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Solutions to Packet

Lesson 1

... minimum is e $= 1/e$ at $t = -\sqrt{2}$

69. $\frac{dy}{dt} = ky$

71. $\frac{dp}{dt} = kp$

73. $f(x) = e^x \Rightarrow f'(x) = e^x \Rightarrow L(x)$ at $x = 0$ is $f'(0)(x - 0) + f(0) = 1(x) + 1 \Rightarrow L(x) = x + 1$

Lesson 2

1. Yes one–to–one, the graph passes the horizontal test
3. Not one–to–one, the graph fails the horizontal test
5. Yes one–to–one, the graph passes the horizontal test
7. $f(x) = x^2 + 1,\ x \ge 0 \Rightarrow y = x^2 + 1.\ \ x = y^2 + 1 \Rightarrow y = \pm\sqrt{x - 1}$ and $(2,5)$ is on the graph of $y = f(x) \Rightarrow$ $(5,2)$ is on the graph of $y = f^{-1}(x) \Rightarrow f^{-1}(x) = \sqrt{x - 1}$.
9. $f(x) = x^3 - 1 \Rightarrow y = x^3 - 1.\ \ x = y^3 - 1 \Rightarrow f^{-1}(x) = \sqrt[3]{x + 1}$.
11. $f(x) = (x + 1)^2,\ x \ge -1 \Rightarrow y = (x + 1)^2$; the inverse, $x = (y + 1)^2 \Rightarrow y + 1 = \pm\sqrt{x} \Rightarrow y = -1 \pm \sqrt{x}$ and $(0,1)$ is on the graph of $y = f(x) \Rightarrow (1,0)$ is on the graph of $y = f^{-1}(x) \Rightarrow f^{-1}(x) = -1 + \sqrt{x}$

Lesson 3

1. a) $e^{\ln 7.2} = 7.2$ b) $e^{-\ln x^2} = \frac{1}{e^{\ln x^2}} = \frac{1}{x^2}$ c) $e^{\ln x - \ln y} = e^{\ln(x/y)} = \frac{x}{y}$

3. a) $2\ln\sqrt{e} = 2\ln e^{1/2} = (2)\left(\frac{1}{2}\right)\ln e = 1$ b) $\ln\left(\ln e^e\right) = \ln(e \ln e) = \ln e = 1$

c) $\ln\left(e^{-x^2 - y^2}\right) = \left(-x^2 - y^2\right)\ln e = -x^2 - y^2$

5. $\ln y = 2t + 4 \Rightarrow e^{\ln y} = e^{2t + 4} \Rightarrow y = e^{2t + 4}$

7. $\ln(y - 40) = 5t \Rightarrow e^{\ln(y - 40)} = e^{5t} \Rightarrow y - 40 = e^{5t} \Rightarrow y = e^{5t} + 40$

9. $\ln(y - 1) - \ln 2 = x + \ln x \Rightarrow \ln(y - 1) = x + \ln x + \ln 2 \Rightarrow \ln(y - 1) = x + \ln 2x \Rightarrow e^{\ln(y-1)} = e^{x + \ln 2x} = e^x e^{\ln 2x} \Rightarrow$ $y - 1 = 2x\,e^x \Rightarrow y = 2x\,e^x + 1$

11. a) $e^{2k} = 4 \Rightarrow \ln e^{2k} = \ln 4 \Rightarrow 2k \ln e = \ln 2^2 \Rightarrow 2k = 2\ln 2 \Rightarrow k = \ln 2$ b) $100\,e^{10k} = 200 \Rightarrow e^{10k} = 2 \Rightarrow$ $\ln e^{10k} = \ln 2 \Rightarrow 10k \ln e = \ln 2 \Rightarrow 10k = \ln 2 \Rightarrow k = \frac{\ln 2}{10}$ c) $e^{k/1000} = a \Rightarrow \ln e^{k/1000} = \ln a \Rightarrow \frac{k}{1000}\ln e =$

Lesson 4

19. $y = \ln 5x \Rightarrow y' = \frac{5}{5x} = \frac{1}{x}$

21. $y = \ln\left(t^2\right) \Rightarrow \frac{dy}{dt} = \frac{2t}{t^2} = \frac{2}{t}$

23. $y = \ln \frac{3}{x} = \ln 3x^{-1} \Rightarrow y' = \frac{-3x^{-2}}{3x^{-1}} = -\frac{1}{x}$

25. $y = \ln(\theta + 1) \Rightarrow \frac{dy}{d\theta} = \frac{1}{\theta + 1}$

27. $y = \frac{x^4}{4} \ln x - \frac{x^4}{16} \Rightarrow y' = x^3 \ln x + \frac{x^4}{4}\left(\frac{1}{x}\right) - \frac{x^3}{4} = x^3 \ln x + \frac{x^3}{4} - \frac{x^3}{4} = x^3 \ln x$

29. $y = t(\ln t)^2 \Rightarrow \frac{dy}{dt} = (\ln t)^2 + (t)(2)(\ln t)\left(\frac{1}{t}\right) = (\ln t)^2 + 2 \ln t$

31. $y = \frac{\ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{\left(\frac{1}{t}\right)(t) - (\ln t)(1)}{t^2} = \frac{1 - \ln t}{t^2}$

33. $y = \frac{\ln x}{1 + \ln x} \Rightarrow y' = \frac{\frac{1}{x}(1 + \ln x) - (\ln x)\left(\frac{1}{x}\right)}{(1 + \ln x)^2} = \frac{\frac{1}{x} + \frac{\ln x}{x} - \frac{\ln x}{x}}{(1 + \ln x)^2} = \frac{1}{x(1 + \ln x)^2}$

35. $y = \ln\left(3t e^{-t}\right) \Rightarrow \frac{dy}{dt} = \frac{3e^{-t} - 3t e^{-t}}{3t e^{-t}} = \frac{1 - t}{t}$

37. $y = \ln\left(\frac{e^\theta}{1 + e^\theta}\right) \Rightarrow \frac{dy}{d\theta} = \frac{\frac{e^\theta\left(1 + e^\theta\right) - \left(e^\theta\right)\left(e^\theta\right)}{\left(1 + e^\theta\right)^2}}{\frac{e^\theta}{1 + e^\theta}} = \frac{\left(e^\theta + e^{2\theta} - e^{2\theta}\right)\left(1 + e^\theta\right)}{\left(1 + e^\theta\right)^2\left(e^\theta\right)} = \frac{1}{1 + e^\theta}$

39. $y = \ln(\csc\theta + \cot\theta) \Rightarrow \frac{dy}{d\theta} = \frac{-\csc\theta\cot\theta - \csc^2\theta}{\csc\theta + \cot\theta} = \frac{(-\csc\theta)(\cot\theta + \csc\theta)}{\csc\theta + \cot\theta} = -\csc\theta$

41. $y = \sqrt{x(x+1)} = (x(x+1))^{1/2} \Rightarrow \ln y = \frac{1}{2}\ln(x(x+1)) \Rightarrow 2\ln y = \ln(x) + \ln(x+1) \Rightarrow \frac{2y'}{y} = \frac{1}{x} + \frac{1}{x+1} \Rightarrow$

$y' = \left(\frac{1}{2}\right)\sqrt{x(x+1)}\left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{2x + 1}{2\sqrt{x(x+1)}}$

43. $y = 3\sqrt{\frac{t}{\ldots}} \quad \left(\frac{t}{\ldots}\right)^{1/2}$